A Wavelet Based Approach to Space-Charge Calculations for Charged Particle Beams

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Abstract

Numerical prediction of the performance of charged particle accelerators is essential for the design and understanding of these machines. Methods to calculate the self-fields of accelerated particles, the so-called space-charge forces, become increasingly important as the demand for high-quality bunches increases. For mesh-based space-charge solvers the knowledge of discontinuities in the particle distribution is important to keep the numerical error small. In this paper we show that the wavelet decomposition technique applied to the space-charge density can detect such discontinuities.

Introduction

Nowadays, charged particle accelerators play an important role for scientific research as well as for medical and industrial applications. They typically accelerate small bunches of charged particles in time varying electromagnetic fields. The calculation of space-charge forces is an important part of the simulation of the behaviour of charged particles in these machines. As the quality of the charged particle bunches increases, so do the requirements for the numerical space-charge calculations. The requirements of new machines like the linear accelerator TTF (TESLA Test Facility) at DESY [2] are for example so tight that effects like non-uniform emission from the photo-cathode and non-cylindrically symmetric fields caused by side-coupled cavities have an effect on the overall performance of the machine. Studying these effects requires 3D calculations with a precision matching the quality of the bunch.

Recently it turned out from simulations for the TTF project that the numerical error of the space-charge calculations is very high at certain edges of the considered bunch. This effect is due to significant changes in the particle distribution. A well-established tool for the detection of such discontinuities is the wavelet decomposition [1,3].

In this paper we investigate the wavelet decomposition of bunches of charged particles. The numerical example shows that the wavelet coefficients provide the required information about discontinuities in the particle distribution. As a numerical test example we have chosen a cigar-shaped bunch with uniform particle distribution, which can be considered as a representative example for bunches generated in the linear accelerator TTF. The objective of this investigation is to improve the accuracy of our recently developed 3D space-charge model in the tracking code GPT [4]. The information from the wavelet decomposition will be exploited to increase both accuracy and stability of the multigrid Poisson solver by choosing better initial mesh line positions.

Calculation of Space-Charge of Charged Particle Bunches

A common method for the calculation of space-charge fields is the particle to mesh method [5]. In order to determine the space-charge fields the potential ϕ is calculated from Poisson's equation

$$\begin{aligned} -\Delta\phi &= \frac{\rho}{\varepsilon_0} \quad \text{in } \Omega \subset R^3, \\ \phi &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where ρ denotes the space-charge density and ε_0 the dielectric constant. The domain Ω is constructed as a box around the bunch. The discretization of Poisson's equation by second order finite

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differences requires an appropriate distribution of mesh lines in every coordinate direction. As a next step the charge of the particles is distributed on the mesh points.

In [7,8] the mesh lines are constructed according to the distribution of the particles. This procedure provides a non-equidistant grid with small step sizes in regions with many particles and big step sizes for regions with only a few particles. For the solution of the resulting system of equations an efficient multigrid algorithm has been developed [7,8].

Wavelet Decomposition of a Charged Particle Bunch

The main idea of a wavelet decomposition is to split a function f (in our case the space-charge density) into a smooth part and details of a certain amplitude, the wavelet part (see for instance [1,3, 10]). This decomposition process is described by a *multiresolution analysis* (MRA). A MRA is defined as a nested sequence of closed subspaces $V_m \subset L^2(R)$ ($L^2(R)$ is the space of square integrable functions) with

$$\{0\} \subset \cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \subset L^2(R)$$

satisfying

$$\overline{\bigcup_{m\in Z} V_m} = L^2(R), \quad \bigcap_{m\in Z} V_m = \{0\}, \quad f(x) \in V_m \Leftrightarrow f(2^m x) \in V_0.$$

Further there exists a function $\varphi \in L^2(R)$, the so-called scaling function, such that the integer translates $\varphi(x-k)\,(k \in Z)$ form a Riezs basis of V_0, that is

$$V_0 = \overline{\text{span}\,\{\varphi(x-k),\, x \in R, k \in Z\}} \quad \text{and} \quad A\sum_{k\in Z} c_k^2 \le \left\| \sum_{k\in Z} c_k \varphi(x-k) \right\|_{L^2}^2 \le B \sum_{k\in Z} c_k^2$$

for all sequences c_k with $\sum_{k\in Z} c_k^2 < \infty$. Furthermore, A and B are positive constants.

Consequently, the space V_m is spanned by a scaled version of φ given by

$$\varphi_{m,k}(x) := 2^{-m/2}\varphi(2^{-m}x - k).$$

The wavelet space W_m is introduced as the orthogonal complement of V_m with respect to V_{m-1}, that is

$$V_{m-1} = V_m \oplus W_m,\; V_m \perp W_m.$$

For every MRA there exists a wavelet ψ, where the translates and dilations given by

$$\psi_{m,k}(x) := 2^{-m/2}\psi(2^{-m}x - k)$$

form an orthonormal basis of W_m.

Assuming the one dimensional function $f \in L^2(R)$ is approximated in V_m by the orthogonal projection $P_m f$ with

$$P_m f = \sum_{k\in Z} c_k^m \varphi_{m,k}.$$

The coefficients c_k^m are determined by $c_k^m = \int_R f \overline{\varphi_{m,k}} \mathrm{d}x$. According to the decomposition of the space V_m with $V_m = V_{m+1} \oplus W_{m+1}$ the projection $P_m f$ can be decomposed as

$$\begin{aligned} P_m f &= P_{m+1} f + Q_{m+1} f \\ &= \sum_{k \in Z} c_k^{m+1} \varphi_{m+1,k} + \sum_{k \in Z} d_k^{m+1} \psi_{m+1,k} , \end{aligned}$$

where $P_{m+1} f$ represents the smooth part of f and $Q_{m+1} f$ represents the details with respect to the space V_m. The coefficients c_k^{m+1} and the wavelet coefficients d_k^{m+1} can be efficiently computed from c_k^m [1]. The simplest example for a scaling function is the 'Haar' function (a step function on the interval [0,1)) with the corresponding 'Haar' wavelet [1]. A further decomposition of $P_{m+1} f = P_{m+2} f + Q_{m+2} f$ gives information about the details on the next lower scale.

Now, the three dimensional case can be considered as a tensor product of the one dimensional decomposition. Denoting with $P_{m,x} f$, $P_{m,y} f$ and $P_{m,z} f$ the approximation with the scaling function in x-, y- and z-direction, respectively. The wavelet parts are analogously referred to as $Q_{m,x} f$, $Q_{m,y} f$ and $Q_{m,z} f$. With this notation the decomposition of $f \in L(R^3)$ (in our application: $f = \rho$) has the form

$$\begin{aligned} P_{m,z} P_{m,y} P_{m,x} f &= P_{m+1,z} P_{m+1,y} P_{m+1,x} f \\ &+ P_{m+1,z} P_{m+1,y} Q_{m+1,x} f + P_{m+1,z} Q_{m+1,y} P_{m+1,x} f + P_{m+1,z} Q_{m+1,y} Q_{m+1,x} f \\ &+ Q_{m+1,z} P_{m+1,y} P_{m+1,x} f + Q_{m+1,z} P_{m+1,y} Q_{m+1,x} f \\ &+ Q_{m+1,z} Q_{m+1,y} P_{m+1,x} f + Q_{m+1,z} Q_{m+1,y} Q_{m+1,x} f . \end{aligned}$$

Here, the representation $P_{m+1,z} P_{m+1,y} P_{m+1,x} f$ is considered as the smooth part of the function f. All other terms with a wavelet part are regarded as the representation of the details. A further decomposition on the next lower scale can be performed with $P_{m+1,z} P_{m+1,y} P_{m+1,x} f$ analogously.

Numerical Results

Electron bunches which are generated in the linear accelerator TTF typically have a cigar shape during acceleration. Therefore, we consider as a representative numerical test case a cylindrical bunch with a length of 0.1 mm and a radius of 0.001 mm. Assuming a uniform particle distribution we have a discontinuity in the particle distribution. Furthermore, the space-charge field is far from linear at the head and tail of the bunch [9]. For the simulations we have chosen a bunch represented by 40,000 sample particles.

The wavelet decomposition has been performed on two scales, beginning with a resolution of 33x33x33 mesh lines for the projection $P_{m,z} P_{m,y} P_{m,x} \rho$. Since the space-charge density ρ is not a smooth function, we have used the Haar wavelet for the decomposition. The wavelet decomposition has been performed with the wavelet toolbox of Matlab [6].

The projection $P_{m,z} P_{m,y} P_{m,x} (-\rho)$ of the cigar shaped bunch is shown in Figure 1. Figure 2 represents the wavelet coefficients from $Q_{m+1,z} P_{m,y} Q_{m+1,x} (-\rho)$. The large wavelet coefficients correspond to the discontinuities in the space-charge distribution. These discontinuities are situated along the radius (left part of the figures) and at the head and the tail of the bunch (right part of the figures). Figure 2 demonstrates that they are clearly detected.

Conclusions

Efficient space-charge calculations with an accuracy matching the quality of an charged particle bunch becomes increasingly important in accelerator design. For the improvement of the numerical errors in simulations it is essential to consider not only the distribution of the particles in a bunch but also discontinuities in the distribution.

We have demonstrated that the wavelet decomposition of the space-charge density is a suitable tool for the detection of such discontinuities. For future work it is planned to exploit the information from the wavelet coefficients for an improved discretization of Poisson's equation.

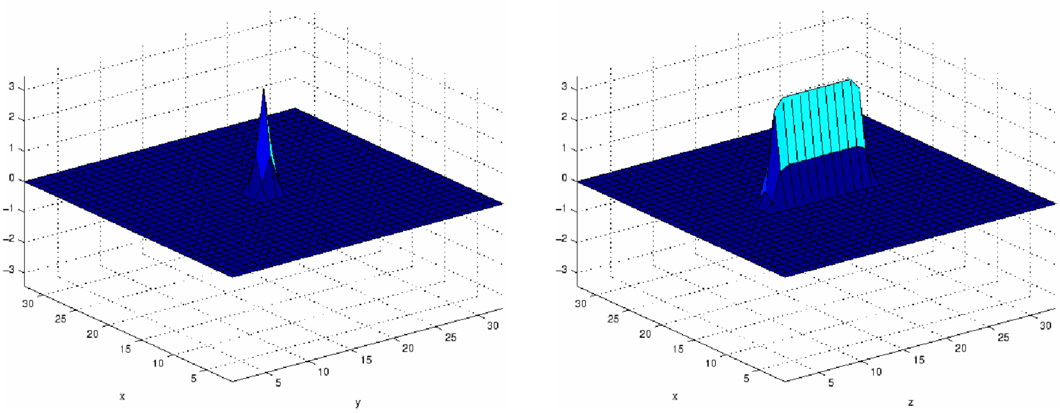

Figure 1. The projection $P_{m,z}P_{m,y}P_{m,x}(-\rho)$ of a cigar shaped bunch shown in the (*x*,*y*)-plane for *z*=17 (left) and in the (*x*,*z*)-plane for *y*=17 (right).

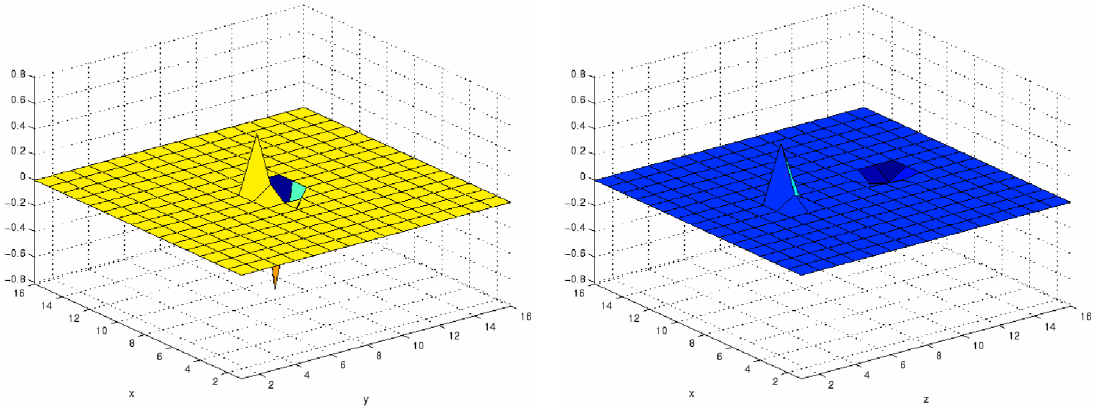

Figure 2. Wavelet coefficients of the projection $Q_{m+1,z}P_{m+1,y}Q_{m+1,x}(-\rho)$ shown in the (*x*,*y*)-plane for *z*=6 (left) and in the (*x*,*z*)-plane for *y*=8 (right).

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