Investigations on the Iterative Solution of Complex Symmetric Systems arising in Electro-Quasistatic

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Abstract

In the electro-quasistatic case Maxwell's equations for time harmonic fields reduce to a complex Poisson's equation and discretization yields a complex symmetric system of equations. We investigate different iterative methods to solve the described non-Hermitian system. The new-tested algorithm GMRES is compared with the proved algorithms COCG and BiCGCR. Additionally the classical preconditioner Jacobi and incomplete LU-decomposition are used. Single eigenvalues are calculated. The numerical experiments originated from problems in high-voltage engineering.

Introduction

An electromagnetic field can be considered as slowly varying if the wavelength is large compared to the problem region which means

$$|kR| << 1 \quad \text{with the wave number } = \omega\sqrt{\mu\varepsilon\left(1 - i\frac{\sigma}{\omega\varepsilon}\right)}$$

and some characteristic dimension R of the studied system. This condition yields for different applications, e.g. we obtain the estimates $|kR| = 2 \times 10^{-6}$ for the following insulator problem or $|kR| = 0.0037$ for 50 Hz fields in the human body. The spatial wavelength is given by $|1/k|$.

For a predominantly electric field, in the case of $|kR| << 1$ it may be assumed $\delta \mathbf{B} / \delta t = 0$ and $\delta \mathbf{D} / \delta t \neq 0$. Under these assumptions (electro-quasistatic (EQS) approximation) a set of simplified Maxwell's equations for time harmonic fields follows:

$$\text{curl}\, \underline{\mathbf{E}} = 0, \tag{1}$$

$$\text{curl}\, \underline{\mathbf{H}} = i\,\omega\, \underline{\mathbf{D}} + \sigma\, \underline{\mathbf{E}} + \underline{\mathbf{J}}_I, \tag{2}$$

$$\text{div}\, \underline{\mathbf{D}} = \rho, \tag{3}$$

$$\text{div}\, \underline{\mathbf{B}} = 0. \tag{4}$$

Therein, $\mathbf{H}$ denotes the magnetic field strength, $\mathbf{E}$ the electric field strength, $\mathbf{D} = \varepsilon\, \mathbf{E}$ the electric field density, $\mathbf{B} = \mu\, \mathbf{H}$ the magnetic induction, $\mathbf{J} = \mathbf{J}_I + \sigma\, \mathbf{E} = \mathbf{J}_I + \mathbf{J}_L$ the current density composed of the conduction current density $\mathbf{J}_I$ and $\mathbf{J}_L$ describing the impressed current density $\mathbf{J}_E$ plus the convection current density. Finally, ρ is the charge carrier density. Here, for a time harmonic field we have used the common representation $\mathbf{E}(r, t) = Re\,(\underline{\mathbf{E}}(\mathbf{r})\, e^{i\,\omega\, t})$ with the complex amplitude $\underline{\mathbf{E}}(\mathbf{r}) = \mathbf{E}\,(\mathbf{r})\, e^{i\phi}$. The non-linear, time-dependent rank two tensors σ (electric conductivity) and ε (permittivity) are assumed to be piecewise constant functions with $\sigma > 0$ and $\varepsilon > 0$ if not stated differently. We assume that appropriate boundary and interface conditions are defined. According to (1) the electric field $\underline{\mathbf{E}}$ is curl-free and thus may be described as the gradient of a scalar potential. Note that this is a *complex* potential: $\underline{\mathbf{E}} = -\,\text{grad}\, \underline{\phi}$. Under these conditions and from (1) - (4) we get the complex divergence equation for the EQS scalar potential

$$\text{div}\,(\,(\,i\,\omega\,\varepsilon + \sigma\,)\,\text{grad}\, \underline{\phi}\,) = \text{div}\,(\,\underline{\mathbf{J}}_I\,). \tag{5}$$

With the Finite Integration Technique (FIT) [1] the continuous equation (5) is transformed into the discretisized ones potential equation

$$\tilde{S}\,(\,i\,\omega\,M_\varepsilon + \mathbf{M}_\sigma)\,\tilde{S}^T\,\phi_E = \tilde{S}\,\mathbf{j}_0$$

with the divergence-operator $\tilde{S}$, the material matrices M_ε, $\mathbf{M}_\sigma$ and the gradient-operator $\mathbf{G} = -\tilde{S}^T$ [2]. The notation $\mathbf{A}_\sigma := \tilde{S}\,\mathbf{M}_\sigma\,\tilde{S}^T$, $A_\varepsilon := \tilde{S}\,M_\varepsilon\,\tilde{S}^T$, $\mathbf{p}_0 := \tilde{S}\,\mathbf{j}_0$, leads to the resulting complex linear system for our EQS calculations

$$(\,\mathbf{A}_\sigma + i\,\omega\,A_\varepsilon)\,\phi_E = p_0\,.$$

The matrix $\mathbf{A} := (\,\mathbf{A}_\sigma + i\,\omega\,A_\varepsilon)$ is a large sparse symmetric matrix with seven bands. It gets almost singular. The large condition number mainly results from large differences in the material parameters.

Applications in High Voltage Engineering

High voltage insulators are stressed by the applied electric field as well as by other environmental factors. As a result of this stress, the surface of the insulating material gets aged and the dielectric material loses its hydrophobic and insulating characteristics. The contamination of the object with water droplets accelerates the aging process. Experimental investigations have shown that with an increase of applied voltage, droplets vibrate first, they are then extended to the direction of the applied electric field and finally flash over bridging water droplets occurs. To improve the understanding of the aging phenomenon it seems advisable to observe single droplets on an insulating surface. The shape of the droplets supplies more information about the status of the insulating material [1]. In addition to the experiments the simulation of the electric field strength near the water droplets is necessary. It allows the calculation of electric forces on the droplet surfaces and thus the droplet movement.

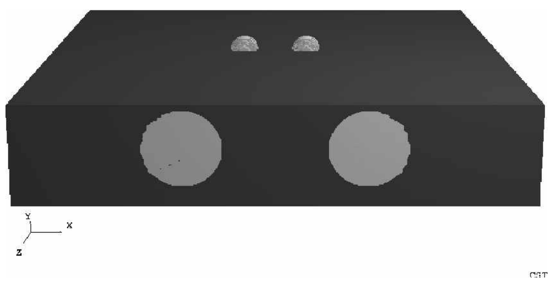

Figure 1 Simplified test specimen, epoxy resin sample with embedded electrodes and two water droplets on it.

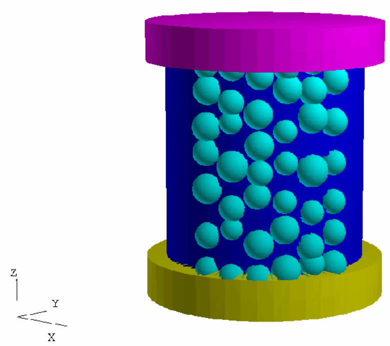

Figure 2. Industrial test specimen, partly covered with single water droplets.

The considered high voltage devices are driven with 50 Hz a.c. voltage, i.e. the electromagnetic field is slowly varying. We model our problem as an electro-quasistatic 3D-problem. The epoxy resin has a relative permittivity of $\varepsilon_r = 4$ and a conductivity of $\sigma = 10^{-12}$ S/m. The water drops have a relative permittivity of $\varepsilon_r = 81$ and a conductivity of $\sigma = 10^{-6}$ S/m. The permittivity of the air surrounding the structure is $\varepsilon_r = 1.000576$. A voltage of 15 kV is used. Example 1 and 2 are a block of epoxy resin with both a length and width of 100mm and a height of 20mm. The test object has horizontally embedded electrodes with a center distance of 35mm and a radius of 7.5mm. We put only two droplets on it with a diameter of 6mm (hemispheres) and a center distance of 10mm according to the accompanying experiments. Example 3 is a cylinder of epoxy resin with a height of 30mm and a radius of 15mm. The electrodes on top and bottom are also shaped cylindrically and have a height of 6mm with an 18mm radius. The droplet radii vary from 1mm to 2.5mm.

Iterative Methods

Iterative methods are tested for the described problem above only in few papers, e.g. in [2,3,4,5]. Our investigations are not a complete summarization. They are to be understood as a supplementation. One important class of iterative methods for solving the complex symmetric system are the Krylov-subspace methods. The aim was the comparison of the proved algorithms COCG and BiCGCR with

GMRES. An explicit description of the applied methods (BiCGCR, COCG, GMRES), references and a detailed investigation may be found in [4] and [5]. We show results for the preconditioners Jacobi and incomplete LU. We tested the incomplete LU with accuracy 1e-1 (LU-1) and 1e-2 (LU-2).

Additionally, we calculated some values of the eigenvalue spectrum. These values are linked up with the global mesh size ratio and the dimension of the systems.

Numerical Results

We tested the introduced algorithms for the two sample objects shown above, the epoxy resin block with two water droplets (see Figure 1) and the cylindrical test specimen with some water droplets (see Figure 2). The typical convergence behavior is shown in the Figures 3 to 6. The equidistant discretization of example 1 leads to a complex symmetric system of equations with 136,160 unknowns. The different preconditioners have an important influence on the accuracy of the residual. Here shows the LU-decomposition with accuracy 1e-2 (LU-2) the best results contrary to the Jacobi preconditioner in former results. Thereby the Jacobi preconditioner requires 0.5 sec, the LU-1 requires 20.7 sec and the LU-2 requires 39.5 sec for precomputing. On the other hand the basic algorithms have especially an influence on the smoothness of the curves.

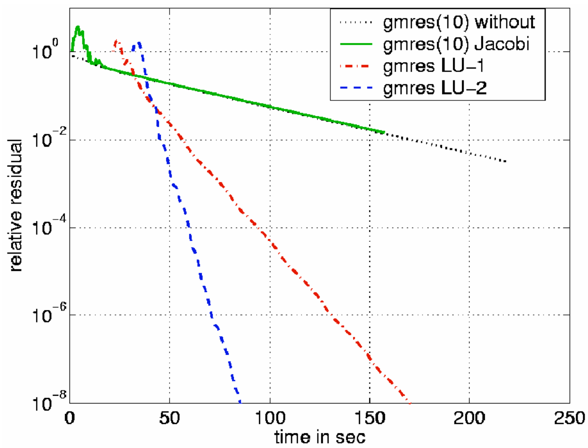

Figure 3. GMRES algorith with different preconditioners

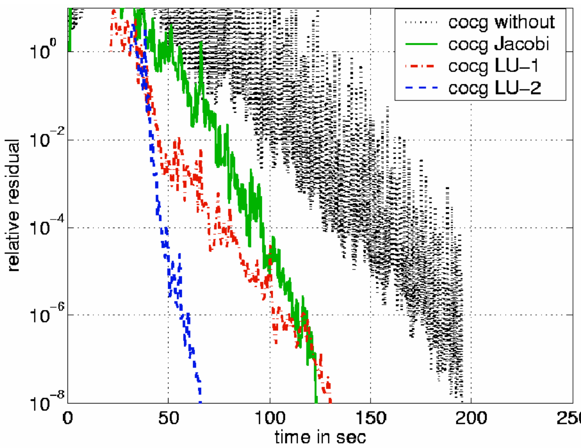

Figure 4. COCG algorithm with different preconditioners

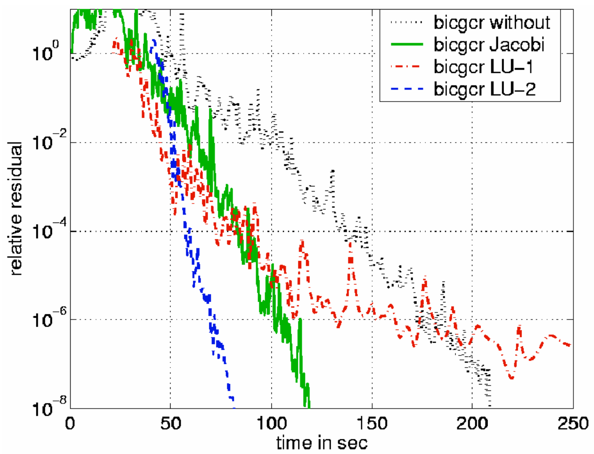

Figure 5. BiCGCR algorithm with different preconditioners

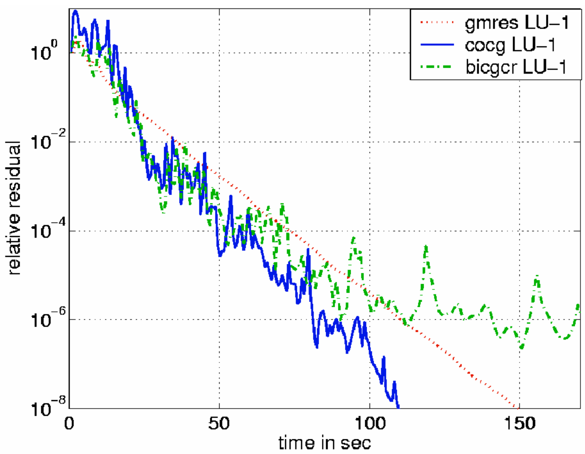

Figure 6. Comparision of the different algorithms

Additionally, we tried to calculate the eigenvalues of our systems. All absolute eigenvalues are smaller than one. It was impossible to calculate all eigenvalues of one system, but single values one can see in Table 1. Here the values with the largest imaginary part, with the smallest real part and with the smallest absolute value are printed. It can be seen that the eigenvalues depend on both, the dimension of the system of equations and the global mesh size ratio. The bad condition of our systems led to problems in the eigenvalue calculation. Thus the computation of the eigenvalues was in every case not successful.

Table 1. Single eigenvalues for different discretizations

	Example 1	*Example 2*	*Example 3*
dimensiom	136,160	45305	101,439
global mesh size ratio	1	1	8
value with larges imaginary part	(0.0863 + 0.0195 i)e-07	(0.8829 + 0.1994 i)e-8	(0.2649 + 0.5977 i)e-10
value with mallest real part		(0.0000 + 0.0003 I)e-9	(0.0041 + 0.0102 i)e-10
smallest absolute value	(0.0001 + 0.3529 i)e-11	(0.0000 + 0.0142 i)e-10	(0.0266 + 0.1024 i)e-10l

Our discretizations are done with the CAE-tool MAFIA. The numerical calculations are done with MATLAB.

Conclusions

We tested three algorithms (GMRES, COCG, BiCGCR) for solving complex symmetric systems of equations in the described technical application. The algorithms COCG and BiCGCR were proved in former calculations. In this paper the GMRES solver is newly tested. The results show that the smoothness depends on the basic algorithm but the fast convergence velocity depends mainly on the preconditioner. It turned out, that GMRES is much smoother than the other algorithms and that the LU-2 preconditioner is the fasted. These conclusions give new hints for solving complex symmetric systems of equations.

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