Modeling the Scattering by Small Holes

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Abstract— The scattering by a hole/aperture within a perfect electric conducing (PEC) plane is a classical electromagnetic problem. As is well known, this scattering can be formulated as the solution of an integral equation where the unknown aperture electric field (or equivalently the magnetic source) is to be retrieved once the incident field is known. When the aperture becomes a hole small in terms of the wavelength, Bethe's diffraction theory [1] or the low frequency approximation (Stevensons series low order terms) [2] can be invoked to approximate the mentioned integral equation. This allows getting analytical results for simple hole's shape, for example in the case of circular hole. For such a case, in this paper, analytical results are compared with numerical simulations obtained by a commercial FDTD forward solver. It is shown that in near zone, low-frequency approximation fits numerical simulations better than the Bethe's theory does. By contrary, Bethe's theory works fairly well in predicting the radiated field in far-zone and allows to obtain a more handling field expression which in principle permits to more easily considering the role of the incident field polarization and the hole's shape. Finally, the case of a hole over the faces of a rectangular waveguide located at different points is analyzed as well. It is found out that Bethe's theory works also in these cases.

1. MATHEMATICAL BACKGROUND

In this section the mathematical notation is introduced and the Bethe's theory as well as the Stevenson approximation are briefly recalled.

Let us consider the scattering experiment described in Figure 1.

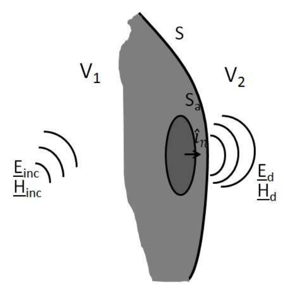

Figure 1: Illustrating the scattering experiment under consideration. A hole in a PEC surface is probed by an impinging field.

A PEC surface divides the whole space in two regions addressed as V_1 and V_2. On such a surface an aperture S_a occurs and the overall structure is illuminated by the impinging field $\underline{H}_{inc}$. The integral equation describing the scattering by such an aperture is:

$$\hat{i}_n \times \underline{H}_0(\underline{r}) = \hat{i}_n \times \iint_{S_a} (\underline{\underline{G}}^{(2)}_{HJ_m}(\underline{r},\underline{r}') + \underline{\underline{G}}^{(1)}_{HJ_m}(\underline{r},\underline{r}')) \cdot \underline{J}_{ms}(\underline{r}')\, d\underline{r}' \qquad \underline{r} \in S_a \tag{1}$$

where $\hat{i}_n$ is the normal vector to the surface pointing from the space V_1 to V_2, $\underline{H}_0$ is the unperturbed magnetic field, (i.e., the solution of Maxwell's equations in absence of the aperture), $\underline{\underline{G}}^{(2)}_{HJ_m}(\underline{r},\underline{r}')$ and $\underline{\underline{G}}^{(1)}_{HJ_m}(\underline{r},\underline{r}')$ are the dyadic Green functions in the space V_2 and V_1, respectively, and $\underline{J}_{ms}$ is the surface density distribution of the equivalent magnetic current related to the electric field $\underline{E}_d$ scattered by the aperture according to the equation:

$$\underline{J}_{ms}(\underline{r}) = -\hat{i}_n \times \underline{E}_d(\underline{r}) \qquad \underline{r} \in S_a \tag{2}$$

Equation (1) was obtained through the equivalent theorem and by enforcing the continuity of tangential components of the electric and magnetic field on the aperture. When S is a plane, image theory allow us simplifying (1) as:

$$\frac{1}{2}\hat{i}_n \times \underline{H}_0(\underline{r}) = \hat{i}_n \times \iint_{S_a} \underline{\underline{G}}^{fs}_{HJ_m}(\underline{r},\underline{r}') \cdot \underline{J}_{ms}(\underline{r}')\, d\underline{r}' \qquad \underline{r} \in S_a \tag{3}$$

where $\underline{\underline{G}}^{fs}_{HJ_m}(\underline{r},\underline{r}')$ is the free space dyadic Green function and $\underline{J}_{ms}$ is doubled with respect to the magnetic current (2).

In [1], when the aperture is small as compared to the wavelength, Equation (3) can be approximated and split in two easier equations such as:

$$-\frac{1}{2}\underline{H}_0\cdot\underline{r} = \iint_{S_a} \eta(\underline{r}')\frac{1}{4\pi\mu|\underline{r}-\underline{r}'|}\,d\underline{r}' \qquad \underline{r}\in S_a \tag{4}$$

$$-\frac{1}{4}\underline{E}_0\times\underline{r} = \iint_{S_a} \underline{J}_{msE}(\underline{r}')\frac{1}{4\pi|\underline{r}-\underline{r}'|}\,d\underline{r}' \qquad \underline{r}\in S_a \tag{5}$$

where η is the surface magnetic charge density that is related to the magnetic currents by the equation:

$$\nabla_t\cdot\underline{J}_{msH}(\underline{r}) = -j\omega\eta(\underline{r}) \tag{6}$$

and $\underline{J}_{msE}$ and $\underline{J}_{msH}$ are the two contributions to the total magnetic currents $\underline{J}_{ms}$.

Furthermore, for the case of circular aperture of radius a, an analytical solution is given as:

$$\underline{J}_{msH}(\underline{r}) = -\frac{4j\omega\mu}{\pi}(a^2-r^2)^{\frac{1}{2}}\underline{H}_0 \tag{7}$$

$$\underline{J}_{msE}(\underline{r}) = \frac{2}{\pi(a^2-r^2)^{\frac{1}{2}}}\underline{r}\times\underline{E}_0 \tag{8}$$

The far field radiated by the first contribution can be approximated as the field radiated by an elementary **magnetic dipole**, whereas the second one as the field radiated by a magnetic loop or equivalently by an **electric dipole** perpendicular to the surface S_a. This is the approximation coming from Bethe's theory.

In [2] a series of integral equations is obtained by expanding Equation (3) in terms of wave number k powers. According to Stevenson, analytical solution for the first two terms of that expansion is given in the case of circular aperture and with an incident plane wave $\underline{E}_i = \underline{\tilde{E}}_i e^{-jk\hat{i}_k\cdot\underline{r}}$:

$$\underline{J}_{ms\rho}^{(0)} = 0 \tag{9}$$

$$\underline{J}_{ms\theta}^{(0)} = \frac{2\rho}{\pi(a^2-\rho^2)^{\frac{1}{2}}}\tilde{E}_{iz} \tag{10}$$

$$\begin{aligned}\underline{J}_{ms\rho}^{(1)} = &-\frac{8j}{3\pi}(\hat{i}_k\cdot\hat{i}_z)(\tilde{E}_{iy}\cos\theta-\tilde{E}_{ix}\sin\theta)(a^2-\rho^2)^{\frac{1}{2}}\\ &-\frac{4j}{3\pi}\left[(\hat{i}_k\cdot\hat{i}_x)\sin\theta-(\hat{i}_k\cdot\hat{i}_y)\cos\theta\right](a^2-\rho^2)^{\frac{1}{2}}\tilde{E}_{iz}\end{aligned} \tag{11}$$

$$\begin{aligned}\underline{J}_{ms\theta}^{(1)} = &\frac{2j}{3\pi}(\hat{i}_k\cdot\hat{i}_z)(\tilde{E}_{ix}\cos\theta+\tilde{E}_{iy}\sin\theta)\left[4(a^2-\rho^2)^{\frac{1}{2}}+\frac{2\rho^2}{(a^2-\rho^2)^{\frac{1}{2}}}\right]\\ &-\frac{4j}{3\pi}\left[(\hat{i}_k\cdot\hat{i}_x)\cos\theta+(\hat{i}_k\cdot\hat{i}_y)\sin\theta\right]\frac{\rho^2+a^2}{(a^2-\rho^2)^{\frac{1}{2}}}\tilde{E}_{iz}\end{aligned} \tag{12}$$

with ρ and θ being the polar coordinates.

2. APERTURES IN PEC PLANE

In Figure 2, it is represented the electric field on an aperture in a PEC plane with a normally incident plane wave, calculated by FDTD software. It can be noted that it is not in agreement with Bethe's solution (7) because the latter is a function that is null on the edge. Instead, results from (9)–(12) are more in agreement since in (12) there is a term that diverges on the edge even in the case of normal incidence.

However, Bethe's theory predicts that the far field scattered by that aperture can be attributed to the field radiated by an elementary magnetic dipole and Figures 4–5 confirm this prediction.

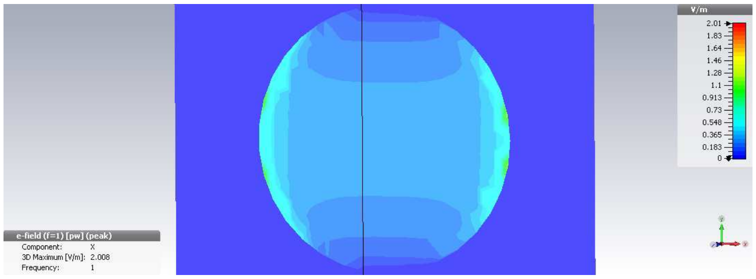

Figure 2: x component of the electric field on the aperture with normal incidence. The radius of the aperture is $\lambda/10$.

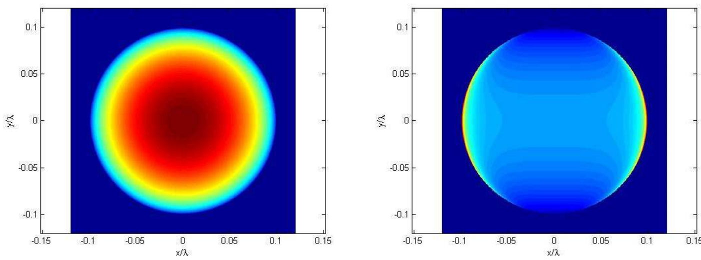

Figure 3: x component of the electric field on the aperture with normal incidence predicted by Equations (9)–(12) (left side) and Equations (7)–(8) (right side). The plots show the electric field for a circle of radius radius $0.99 \times \lambda/10$ to avoid the singularity in Equation (12). By comparison with Figure 2, it can be argued that Stevenson's theory works better than Bethe's theory for the near zone field.

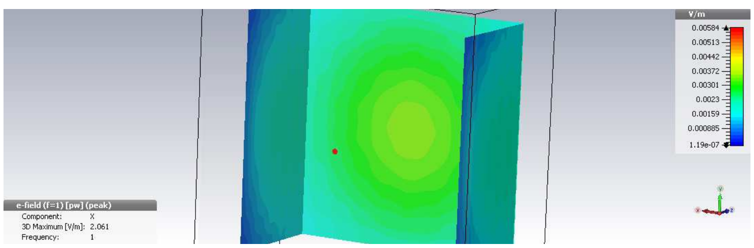

Figure 4: x component of the electric field scattered by an aperture on a PEC plane with normal incidence. The radius of the aperture is $\lambda/10$ and the field is plotted on 3 surfaces: one parallel to the PEC plane distant 5λ from it and the other two perpendicular to it distant 4λ from the aperture.

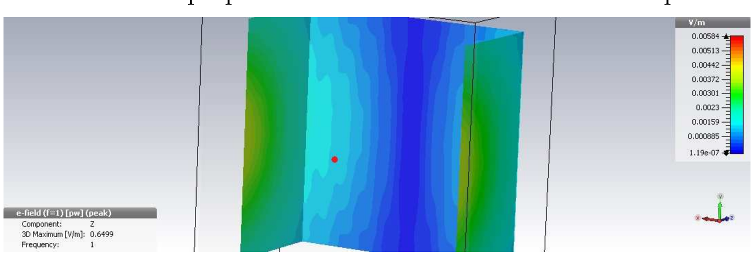

Figure 5: z component of the electric field scattered by an aperture on a PEC plane with normal incidence and in the same configuration as in Figure 4.

3. APERTURES IN RECTANGULAR PEC WAVEGUIDE

Although Bethe's theory was developed from the integral equation relative to a CEP plane, it is reasonable to check if it works for apertures made in more complex PEC structures. Here, we consider the hole occurring in rectangular waveguide still by considering the apertures being small with respect to the structure and the wavelength. The following figures of the fields scattered outside the guide are obtained thanks to a FDTD software. Figure 6 shows the amplitude of y component of the electric field radiated by an aperture in the centre of the waveguide's upper surface over three different plane located at 1.5, 3.5 and 5.5 times the wavelength from the waveguide's upper surface. In this case the unperturbed field $\underline{E}_0$ and $\underline{H}_0$ corresponds to the waveguide fundamental mode and are directed respectively along y and x directions. As can be noted, the scattered field

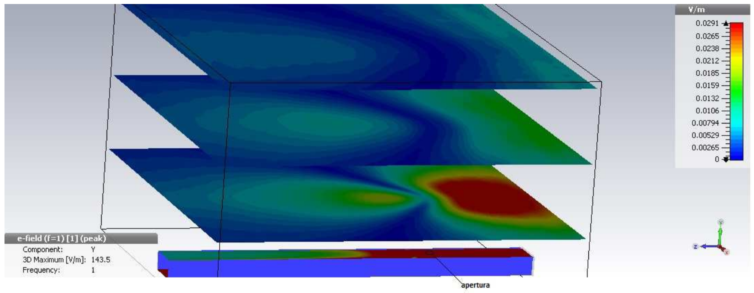

Figure 6: y component's amplitude of electric field radiated by an aperture in the centre of rectangular waveguide's upper surface. The waveguide (in the lower side of the figure) has a section wich dimensions are $1\lambda \times 0.5\lambda$, whereas the aperture's radius is $\lambda/20$.

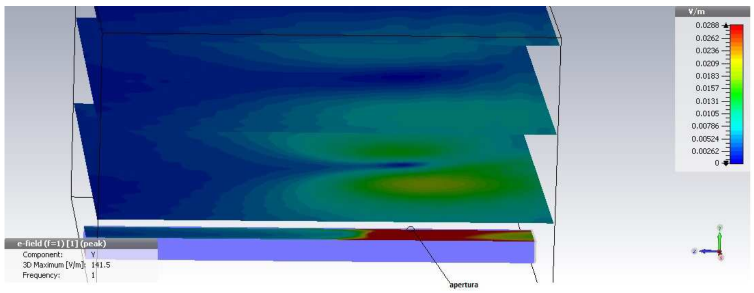

Figure 7: y component's amplitude of electric field radiated by an aperture near the edge of rectangular waveguide's upper surface. The dimensions are the same as in the Figure 6. The aperture's centre is $\lambda/10$ from the upper surface's edge.

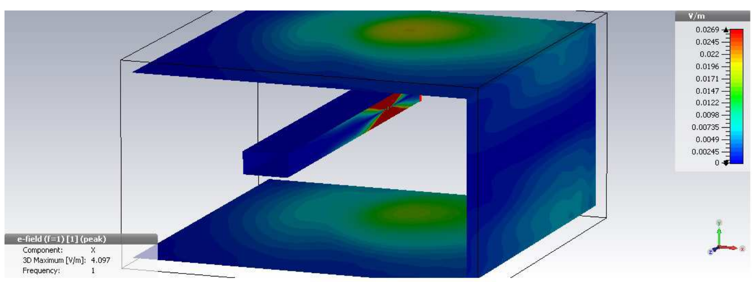

Figure 8: x component's amplitude of electric field radiated by an aperture on rectangular waveguide's side surface. The dimensions are the same as in the Figure 6.

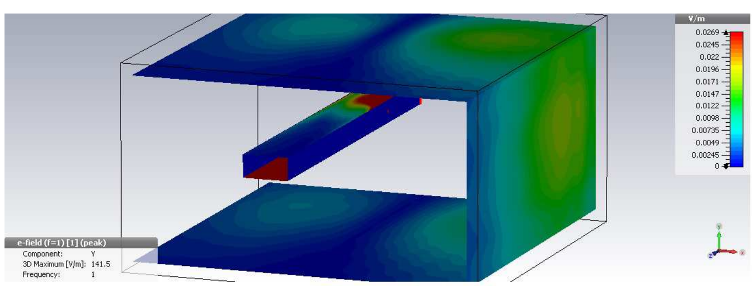

Figure 9: y component's amplitude of electric field radiated by an aperture with the same configuration as in Figure 8.

is similar to that radiated by a Huygens source radiating toward negative direction of z axis. That is in strict accordance to the Bethe's theory.

Figure 7 reports the scattered field when the aperture is moved near the edge of the upper waveguide surface. In this case $\underline{H}_0$ has a z component, so the Huygens source is rotated and the propagation's direction is mainly along the x axis. Once again in accordance to the Bethe's theory.

If the aperture is placed on the lateral waveguide surface, $\underline{E}_0$ is equal to zero and $\underline{H}_0$ is directed along z direction. So the field radiated is that one of an elementary magnetic dipole directed along z (see Figures 8–9).

4. CONCLUSION

The scattering by small holes in PEC surface have been dealt with. Two approches have been shown, one developed by the german physicist Hans Albrecht Bethe and one based on series expansions. Numerical simulations of far-field have shown great agreement with Bethe's theory, but this fails to predict the near-field. For the last the "low frequency expansion" theory is more adequate.

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