Calculations of Inductance and Induced EMF in a Planar Pickup Coil

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Abstract— We present derivations of the equations and numerical solutions for the inductance and induced emf in a rectangular pickup loop in the presence of a current-carrying short straight wire. We explore three orientations of the loop with respect to the wire: in the plane of the wire with one edge of the loop parallel to the wire, in the plane of the wire with one edge of the loop at an arbitrary angle with respect to the wire, and with the loop at an arbitrary angle in three-space with respect to the wire. Solutions in closed-form are presented for the simple cases. This work has applications related to pickup coils in general.

1. INTRODUCTION

Electromagnetic calculations involving the ideal situations typically make for easier calculation that are covered in most introductory textbooks [1, 2], but are usually not indicative of practical, working conditions of a device. In more complex configurations the equations typically take closed-form solutions and approximations for specific geometrical shapes and arrangements of conductors [3, 4] but integral equations are not usually given.

For example, consider the calculation of the flux through a rectangular loop adjacent to the an infinitely long, current carrying straight wire. The rectangular loop acts as a pick-up coil since it is flux-linked to the straight wire. By making the straight wire infinitely long, the calculation is easier because of the constant flux inside an infinitesimally thin rectangular box that has its long side parallel to the wire. Consider the arrangement shown in Figure 1 (with the wire's z axis infinite in both directions). Using the dimensions shown in Figure 1, the flux is easily given by the integral

$$\Phi = \int_a^{a+b} \int_0^c \frac{\mu_0 I}{2\pi y} dz dy = \frac{\mu_0 I c}{2\pi} \ln\left[1 + \frac{b}{a}\right]. \tag{1}$$

However, the problem becomes more difficult when the infinitely long restriction is lifted since the magnetic field is no longer symmetric in the rectangular loop. Additionally, the orientation of the loop may vary with respect to the wire. In these cases, finding closed-form algebraic expressions are typically not practical and numerical solutions are usually sufficient. In this article we show how the induced emf can be calculated for various geometries for this wire-loop combination.

2. DERIVATION I: SIMPLIFIED GEOMETRY OF A PLANAR LOOP

Consider the single-turn rectangular loop parallel to the straight wire as shown in Figure 1. The straight wire carries a time varying current $I(t)$ and the magnetic field intensity at locations x and y and time t is given by the expression

$$\vec{B}(y, z, t) = \frac{\mu_0 I(t)}{4\pi y} \left[\frac{(z - L)}{\sqrt{y^2 + (z - L)^2}} - \frac{z}{\sqrt{y^2 + z^2}}\right] (-\hat{\imath}) \tag{2}$$

where y and z are the coordinates measured from the bottom of the wire (taken as the origin). The expression, integrated over the area of the loop $[d\vec{A} = dydz(-\hat{\imath})]$, is the time rate of change of flux $\Phi(z, t)$ and is

$$\Phi(z, t) = \frac{\mu_0 I(t)}{4\pi} \int_{z_0}^{z_0+c} \int_a^{a+b} \left[\frac{(z - L)}{y\sqrt{y^2 + (z - L)^2}} - \frac{z}{y\sqrt{y^2 + z^2}}\right] dydz. \tag{3}$$

This equation can be written as

$$\Phi(z_0, t) = \frac{\mu_0 I(t)}{4\pi} \left[\bar{I}(z_0 - L) - \bar{I}(z_0)\right] \tag{4}$$

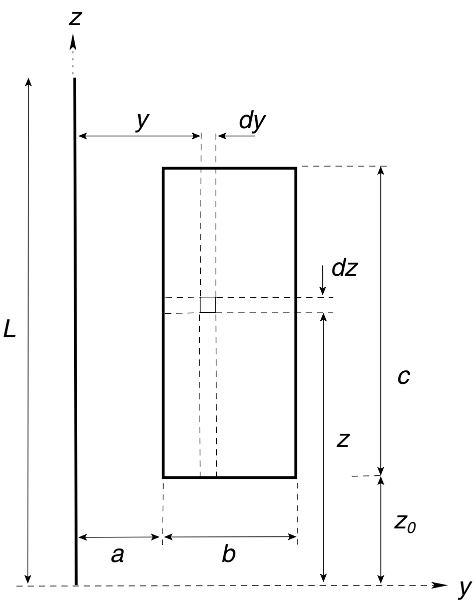

Figure 1: Configuration of loop parallel to straight wire.

where the integrals differ only in the replacement of $z_0 - L$ for z_0 which simplifies the closed-form solution to

$$\begin{aligned}\bar{I}(z_0) \;=\; & \sqrt{(a+b)^2+(z_0+c)^2}-\sqrt{a^2+(z_0+c)^2} \\ & -(z_0+c)\ln\frac{a}{a+b}\frac{(z_0+c)+\sqrt{(a+b)^2+(z_0+c)^2}}{(z_0+c)+\sqrt{a^2+(z_0+c)^2}} \\ & -\sqrt{(a+b)^2+z_0^2}+\sqrt{a^2+z_0^2}+z_0\ln\frac{a}{a+b}\frac{z_0+\sqrt{(a+b)^2+z_0^2}}{z_0+\sqrt{a^2+z_0^2}} \end{aligned} \tag{5}$$

with the expression for $\bar{I}(z_0 - L)$ given as the same expression above but with $z_0 - L$ substituted for z_0. While the above expression does show that the time-varying flux can be expressed in closed-form, its actual use in calculations can be un-wielding. It can be shown that in the proper limits, this expression reduces to Equation (1). Since numerical results are sufficient for experimental comparisons, we include the above expression only for completeness and will use only integral expressions for the rest of our calculations.

Since mutual inductance is defined as $M \equiv \Phi/I$ the expression for the mutual inductance between the straight wire and the rectangular loop is $M = \mu_0/4\pi[\bar{I}(z_0 - L) - \bar{I}(z_0)]$ where it is noted that the integral expressions contain only geometric terms, consistent with the fact that inductance is determined only by geometry. Using Faraday's law, the induced emf that appears as $V(t)$ across the terminals of the loop is very closely approximated at $V(t) = -\dot{\Phi} = -M\dot{I}(t)$ when the gap in the loop is very small.

Consider a sinusoidal time-varying current in the straight wire $I(t) = I_0 \cos\omega t$ where I_0 is the maximum current in the wire and $\omega = 2\pi f$ is the angular frequency. The instantaneous value of the induced emf is $V(t) = \omega M I_0 \sin\omega t$. Since the *rms* value is what is physically measurable it is easily calculated using the time-average integral over one period for the previous expression

$$V_{rms} = \frac{\mu_0 \omega I_0}{8\pi}\left[\bar{I}(z_0 - L) - \bar{I}(z_0)\right] \tag{6}$$

where $[\bar{I}(z_0 - L) - \bar{I}(z_0)]$ is numerically calculated for various loop-wire geometries.

3. DERIVATION II: GENERAL CASE OF A PLANAR LOOP AT ANGLE θ

Consider the single-turn rectangular loop shown in Figure 2(a). In this configuration the loop's angle with respect to the finite straight wire is given by the parameter θ (measured parallel to the length of the finite wire). We first note that there is a critical angle for the loop's orientation with

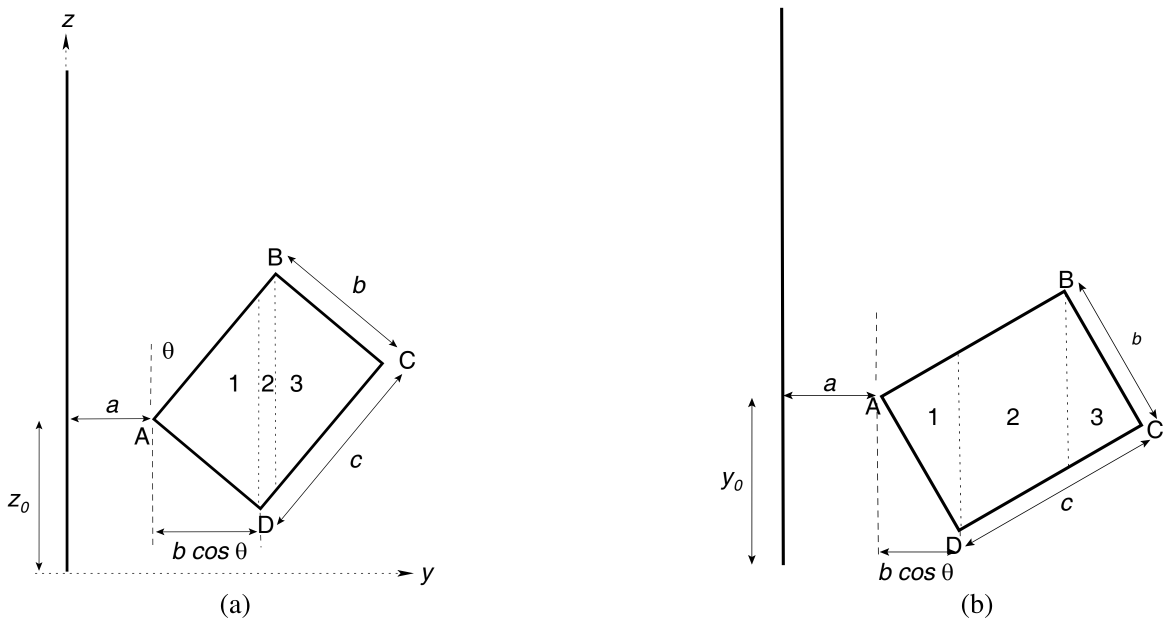

Figure 2: Configuration of straight wire with loop at angle (a) less than the critical angle and (b) greater than the critical angle.

respect to the straight wire. It occurs when a line drawn from the corner angle A to corner angle B is perpendicular to the straight wire. The critical angle is then given by $\tan\theta_c = b/c$. This will split the rectangle into two equal triangles with a common hypotenuse and integration over the loop is a little more straightforward. However, at any other angle there will be three regions to integrate over (two triangular regions and the parallelogram region between them as seen in the figure and labeled as 1, 2, and 3).

There are two cases to consider. One is when the loop is at an angle less than the critical angle θ_c and the other is when it is greater than the critical angle. The case for when the loop is exactly at the critical angle is then either of these cases and both results are the same when evaluated at this critical angle.

Angle less than θ_c

The first case we consider is when the angle is less than the critical angle. The three regions of the loop lend themselves to three separate integrals for $I(\theta < \theta_c) = I_1 + I_2 + I_3$ which gives

$$\begin{aligned} I(\theta) \;=\; & \int_a^{a+c\sin\theta} \int_{z_0-(x-a)\tan\theta}^{z_0+\frac{y-a}{\tan\theta}} B(y,z,t)dydz + \int_{a+c\sin\theta}^{a+b\cos\theta} \int_{z_0-(c\sin\theta+y)\tan\theta}^{z_0+c\cos\theta-y\tan\theta} B(y,z,t)dydz \\ & + \int_{a+b\cos\theta}^{a+b\cos\theta+c\sin\theta} \int_{z_0+\alpha-\frac{c\sin\theta-(y-a-b\cos\theta)}{\tan\theta}}^{z_0+\alpha+[c\sin\theta-(y-a-b\cos\theta)]\tan\theta} B(y,z,t)dydz \end{aligned} \tag{7}$$

where $\alpha = c/\cos\theta - b\sin\theta - c\sin\theta\tan\theta$.

Angle greater than θ_c

The second case we consider is when the angle is greater than the critical angle. The three regions of the loop for giving $I(\theta > \theta_c) = I_1 + I_2 + I_3$ then yields

$$\begin{aligned} I(\theta) \;=\; & \int_a^{a+b\cos\theta} \int_{z_0-(x-a)\tan\theta}^{z_0+\frac{y-a}{\tan\theta}} B(y,z,t)dydz + \int_{a+b\cos\theta}^{a+c\sin\theta} \int_{z_0-(b\sin\theta)+\frac{y-a-b\cos\theta}{\tan\theta}}^{z_0+\frac{y-a}{\tan\theta}} B(y,z,t)dydz \\ & + \int_{a+c\sin\theta}^{a+c\sin\theta+b\cos\theta} \int_{z_0+\alpha\frac{b\cos\theta-(y-a-c\sin\theta)}{\tan\theta}}^{z_0+\alpha+[b\cos\theta-(y-a-c\sin\theta)]\tan\theta} B(y,z,t)dydz \end{aligned} \tag{8}$$

where now $\alpha = (c - b\tan\theta)\cos\theta$ and the total integration giving $I(\theta > \theta_c) = I_1 + I_2 + I_3$.

4. DERIVATION III: ARBITRARY ORIENTATION

Consider the single-turn rectangular loop shown in Figure 3. The single loop is located in the (x, y, z) coordinate system with integration carried out over the (y, z) plane of the loop. The finite wire is oriented along z' axis of a coordinate system that is rotated about the (x, y, z) axes with the two origins coincident.

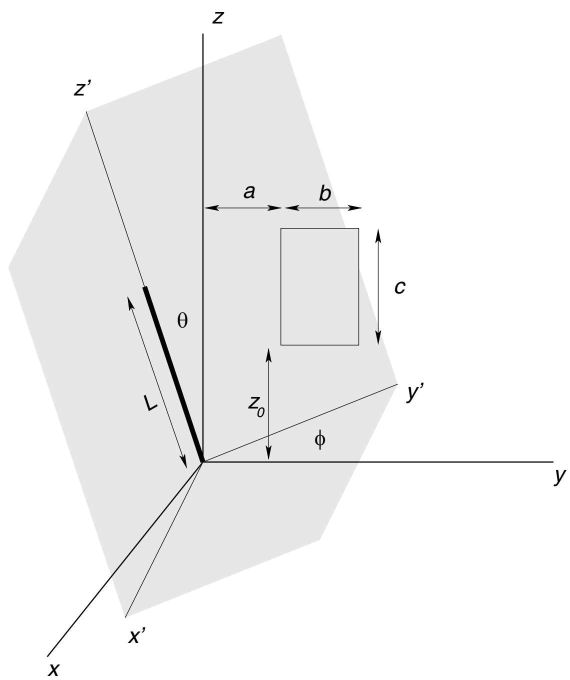

Figure 3: General rotation figure.

Using the Biot-Savart Law, the integral in the primed coordinate system

$$\vec{B}(\vec{r'}, t) = \frac{\mu_0 I(t)}{4\pi\epsilon_0} \int_0^L \frac{\left(y'\hat{i'} - x'\hat{j'}\right) d\zeta'}{\left(x'^2 + y'^2 + (z' - \zeta)^2\right)^{3/2}} \tag{9}$$

can be solved in a straightforward manner to yield

$$\vec{B}(\vec{r'}, t) = \frac{\mu_0 I(t)}{4\pi\epsilon_0} \frac{(y'\hat{i'} - x'\hat{j'})}{(x'^2 + y'^2)} \left[\frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} - \frac{z' - L}{\sqrt{x'^2 + y'^2 + (z' - \zeta)^2}} \right] \tag{10}$$

for a closed-form solution of the magnetic field. Now, the integration to determine the flux is easier to carry out over the unprimed coordinate system,

$$\Phi(t) = \int_a^{a+b} \int_{z_0}^{z_0+c} \vec{B}(\vec{r}, t) \cdot d\vec{A} \tag{11}$$

but this requires the magnetic field equation to map from $\vec{B}(\vec{r'}, t)$ to $\vec{B}(\vec{r}, t)$. This is easily accomplished using the transformation equation $\mathbf{x} = \boldsymbol{\lambda}\mathbf{x}'$ where the transformation matrix $\boldsymbol{\lambda}$ transforms coordinates in the primed frame $\mathbf{x}'$ to those in the unprimed frame $\mathbf{x}$ using the matrix equation

$$\mathbf{x_i} = \sum_{i,j} \lambda_{i,j} \mathbf{x'_i} \tag{12}$$

where the elements of matrix (λ_{ij}) are the direction cosines of the $x'_i - x_j$ axis (i.e., $\lambda_{ij} = \cos(x'_i, x_j)$). Additionally, the transformation matrix is used to transform the unit vectors from the primed to unprimed coordinate system therefore allowing the complete transformation $\mathbf{B}(\vec{r'}, t) \mapsto \mathbf{B}(\vec{r}, t)$ that is given by

$$\begin{aligned}\mathbf{B}(\vec{r}, t) = &\frac{\mu_0 I(t)}{4\pi\epsilon_0}\frac{1}{(\lambda_{12}y+\lambda_{13}z)^2+(\lambda_{22}y+\lambda_{23}z)^2}\\ &\times\left[\frac{(\lambda_{32}y+\lambda_{33}z)}{\sqrt{(\lambda_{12}y+\lambda_{13}z)^2+(\lambda_{22}y+\lambda_{23}z)^2+(\lambda_{32}y+\lambda_{33}z)^2}}\right.\\ &\left.-\frac{(\lambda_{32}y+\lambda_{33}z-L)}{\sqrt{(\lambda_{12}y+\lambda_{13}z)^2+(\lambda_{22}y+\lambda_{23}z)^2+(\lambda_{32}y+\lambda_{33}z-L)^2}}\right]\\ &\times\left[\lambda_{11}(\lambda_{22}y+\lambda_{23}z)-\lambda_{21}(\lambda_{12}y+\lambda_{13}z)\right]. \end{aligned} \tag{13}$$

With the appropriate direction cosines, this equation is substituted into Equation (11) and the flux integral carried out over $d\vec{A} = dydz\hat{i}$, which can then be used to calculate the mutual inductance and induced emf in the loop.

5. CONCLUSION

We have derived numerical expressions for the mutual inductance and emf for a pick-up coil in the presence of a current-carrying finite length wire. Three geometries were considered and typical numerical results were presented. The expressions lend themselves to easy implementation by programs such as Mathcad, Mathematica, or other programming languages.

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