

Dynamical and Stochastic Approach to Non-linear Polarization Optics

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Abstract— We develop the dynamical as well as stochastic aspects for the polarized light transmitting the non-linear media. The central concept is the non-linear birefringence, which is realized for the two-component non-linear Schrödinger equation (NLSE). We put an ansatz: $\psi = (\psi_1, \psi_2) = (a(z), b(z))F(\mathbf{x})$, where $F(\mathbf{x})$ is the single component scalar wave, which is given as the “soliton” solution for the NLSE. The parameter z is the coordinate of propagation direction, and the $\mathbf{x} = (x, y)$ is that of polarized plane. By substituting this ansatz into the action function $S = \int \psi^* (i\lambda \frac{\partial}{\partial z} - \mathcal{H}) \psi d\mathbf{x} dz$, where λ is the wave length for traveling wave, and \mathcal{H} is the Hamiltonian which is written by 2×2 matrix, and using the variation principle, we derive the coupled equation for $a(z)$ and $b(z)$, which can be reduced to the equation of motion for the Stokes parameters. Taking into account of randomness and dissipation contained in the birefringent media, we derive the Langevin equation for the Stokes parameters. From this equation, we can obtain the Fokker-Planck (FP) equation for the probability distribution with respect to Stokes parameters. In order to derive the FP equation, we employ the technique of functional integral, which is based on the assumption of the Gaussian white noise for the random fluctuation arising from birefringence.

1. INTRODUCTION

The polarization is a characteristic property inherent in transversal nature of light wave (or electromagnetic field). This gives rise to a variety of phenomena when the light transmits optically anisotropic medium [1, 2]. The anisotropy of medium is complement to polarization degree of freedom and this complementary nature provides a strong motivation to investigate the evolutionary behavior of polarized light in various linear as well as non-linear optical substances (see, e.g., [3–5]).

The standard procedure to describe the evolution of polarization is based on *para-axial scheme* [3–5] (assisted by an envelope approximation). This describes the light propagation along a prescribed direction that is perpendicular to the polarization plane. Two polarization components are mixed each other owing to the dielectric tensor which causes a polarization change. The central concept to describe the change of polarization is the Stokes parameter.

From the applicational point of view, there have been a great deal of studies of polarization phenomena from the microscopic level to cosmological scale. Especially mentioned is that the recent development in the observations of the cosmic microwave background (CMB) polarization, which may connect with the substructure of the Universe [6, 7]. In addition, the characteristic polarization mode, so-called “B-mode”, which are generated by gravitational lens effect [8] and primordial gravitational waves [9, 10], is hot topic in cosmology and astrophysics.

What we are going to develop in what follows is concerning the dynamical as well as stochastic aspect for the polarized light transmitting the non-linear media. We need to develop the stochastic aspect, because it plays a crucial role in actual problems concerning the birefringent phenomena. This can be realized as the dynamics of the Stokes parameters, which is reduced to the Fokker-Planck (FP) equation [11, 12] via the Langevin equation by using the functional integral. We expect this attempt may provide a possible basis for problems of CMB polarization above mentioned.

2. FIELD EQUATION FOR THE POLARIZED LIGHT

The polarization characterizes the light propagation in anisotropic materials. This is orthogonal to traveling direction and is described by two component (complex) wave function.

We start with the brief sketch for the two components non-linear Schrödinger equation (NLSE) for the light wave traveling through the non-linear substance [13–15]. We put electric field as

$$\mathbf{E}(\mathbf{x}, z) = \mathbf{f}(\mathbf{x}, z) e^{ikn_0 z} \quad (1)$$

where n_0 refractive index in isotropic media, and $\mathbf{f}(\mathbf{x}, z)$ is a modified wave function. The parameter z is the coordinate of propagation direction, and the $\mathbf{x} = (x, y)$ is the coordinate of polarized plane. The $k = \frac{\omega}{c}$, and $n_0 (\equiv \sqrt{\varepsilon_0})$ means the refractive index for the case as if the medium is isotropic.

The amplitude $\mathbf{f}(\mathbf{x}, z)$ is written as $\mathbf{f} = {}^t(f_1, f_2) = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$, where \mathbf{e}_1 and \mathbf{e}_2 denotes the basis of linear polarization.

Here we adopt the para-axial approximation [5, 15]: We assume that \mathbf{f} is slowly varying function of z , that means $|\frac{\partial^2 \mathbf{f}}{\partial z^2}| \ll k|\frac{\partial \mathbf{f}}{\partial z}|$. The circular basis instead of the linear polarization ($\mathbf{e}_1, \mathbf{e}_2$), which is written as

$${}^t(\mathbf{e}_+, \mathbf{e}_-) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \cdot {}^t(\mathbf{e}_1, \mathbf{e}_2) \equiv T^t(\mathbf{e}_1, \mathbf{e}_2) \quad (2)$$

By introducing the wave function

$$\psi = T\mathbf{f} = {}^t(\psi_1, \psi_2), \quad (3)$$

we have the two component Schrödinger-type equation for ψ :

$$i\lambda \frac{\partial \psi}{\partial z} = \mathcal{H}\psi \quad (4)$$

with the transformed ‘‘Hamiltonian’’

$$\mathcal{H} = -\frac{\lambda^2}{2n_0} \nabla_{\perp}^2 + V. \quad (5)$$

where λ is wave length of the polarized wave, $\nabla_{\perp}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ means the Laplacian respect to the polarized plane. and $V = V_{\text{NL}} + V_{\text{L}}$ (V_{NL} and V_{L} represent potential of non-linear and linear, respectively) is defined as following:

$$V_{\text{NL}} = \frac{g_0}{2} |\psi|^2 + g \begin{pmatrix} -\frac{|\psi_1|^2 - |\psi_2|^2}{2} & \frac{\psi_2^\dagger \psi_1}{2} \\ \frac{\psi_1^\dagger \psi_2}{2} & \frac{|\psi_1|^2 - |\psi_2|^2}{2} \end{pmatrix}, \quad V_{\text{L}} = \begin{pmatrix} \gamma & \alpha - i\beta \\ \alpha + i\beta & -\gamma \end{pmatrix} \quad (6)$$

Here, the first term of V_{NL} describes the scalar type non-linear interaction with the coupling constant g_0 , whereas the second term represents the non-linear birefringence with g [16, 17]. The parameters α , β and γ in V_{L} are described by the effects of external fields, for example, electric and magnetic fields.

3. THE EVOLUTION OF THE STOKES PARAMETERS

In the following, we consider a specific solution of the two-component NLSE such that the solution incorporates the polarization degree of freedom in a simple way. In order to realize this we put ansatz for ψ by using the spinor form:

$$\psi = F(\mathbf{x}) \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = F(\mathbf{x}) \begin{pmatrix} \cos \frac{\theta(z)}{2} \\ \sin \frac{\theta(z)}{2} e^{i\phi(z)} \end{pmatrix} \quad (7)$$

where (θ, ϕ) represents the angular coordinate characterizing the polarization, namely, that of the Poincare Sphere, and $F(\mathbf{x})$ is the single component scalar wave which becomes the ‘‘soliton’’ solution of NLSE;

$$\left[-\frac{\lambda^2}{2n_0} \nabla_{\perp}^2 + \frac{g_0}{2} |\psi|^2 + k \right] F(\mathbf{x}) = 0$$

We substitute this ansatz into the action function;

$$S = \int \psi^\dagger \left(i\lambda \frac{\partial}{\partial z} - \mathcal{H} \right) \psi d^2x dz = \int \mathcal{L} dz \quad (8)$$

where the \mathcal{L} is the Lagrangian. By using the variation principle $\delta S = 0$, we can obtain the coupled equation for $a(z)$ and $b(z)$, which is reduced to the equation of motion for the Stokes parameters. The Stokes parameters are defined as

$$\mathbf{S}_i = \psi^\dagger \sigma_i \psi, \quad \mathbf{S} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (9)$$

Then we can get the equation satisfied by Stokes variables

$$\frac{d\mathbf{S}}{dz} = -\mathbf{S} \times \frac{\partial H}{\partial \mathbf{S}} \quad (10)$$

where H is the Hamiltonian; $H = \psi^\dagger \mathcal{H} \psi$. This is just an analogy with the equation of motion of real spin. The concrete expression of H is given as

$$H = \frac{A}{\lambda} [\alpha S_x + \beta S_y + \gamma S_z] - \frac{1}{2} B (2S_z^2 - 1) - C \quad (11)$$

where the parameters A , B and C are defined by the soliton solution $F(\mathbf{x})$.

The next problem takes into account of randomness contained in the birefringence media. Besides the random fluctuations, we have to take into account of an another effect coming from dissipation, that is caused by the inevitable effect of absorption. This can be given in a phenomenological form; $\frac{\partial H}{\partial \mathbf{S}} \rightarrow \frac{\partial H}{\partial \mathbf{S}} + \mu \frac{d\mathbf{S}}{dz}$. Thus we have

$$\frac{d\mathbf{S}}{dz} = \frac{1}{1 + \mu^2} \left[\mu \frac{\partial H}{\partial \mathbf{S}} - \mu \left(\mathbf{S} \cdot \frac{\partial H}{\partial \mathbf{S}} \right) - \mathbf{S} \times \frac{\partial H}{\partial \mathbf{S}} \right] = -\mathbf{A} \quad (12)$$

This is an analogy of the ‘‘Landau-Lifschitz equation’’, that is well known in the ferromagnetic theory [18].

An example — We assume the non-dissipation state ($\mu = 0$) and $\alpha = \beta = 0$, we can simply derive the equation for the Stokes parameters:

$$S_x = S_0 \sin \zeta \cos(Dz), \quad S_y = S_0 \sin \zeta \sin(Dz), \quad S_z = S_0 \cos \zeta \quad (13)$$

where the parameter ζ is the angle respect to S_z , and $D = \frac{A}{\lambda} \gamma - B S_z$. This represents the uniform rotation of the elliptic polarization characterized by the constant angle ζ . This can be regarded as a generalization of the Faraday effect.

4. THE LANGEVIN EQUATION AND FUNCTIONAL INTEGRAL

In what follows we have a mind of the case that the fluctuation effect gives rise to the randomness for the ‘‘pseudo-magnetic field’’, which is written as ξ , so we replace $\frac{\partial H}{\partial \mathbf{S}} \rightarrow \frac{\partial H}{\partial \mathbf{S}} + \xi$. Here $\xi(z)$ is assumed to be the Gaussian white noise as a random magnetic field. Hence we have

$$\frac{d\mathbf{S}}{dz} = -\mathbf{A} + \eta \quad (14)$$

Here η stands for a random torque, $\eta = \mathbf{S} \times \xi$. From this Langevin equation, we can derive the FP equation for the probability distribution respect to Stokes parameter. In order to get the FP equation, we employ the technique of functional path integral.

As a result of the randomly uncorrelated nature of the function ξ , η can be also expected to obey the Gaussian white noise, which is expressed as $\langle \eta_i(z) \rangle = 0$, $\langle \eta_i(z) \eta_j(z+u) \rangle = h \delta_{ij} \delta(u)$ with $\delta(u)$ is the delta function, and the h means the diffusion constant. It is assumed that $\langle \eta^2 \rangle = 2h$, and its probability distribution may be given by the standard Gaussian functional form:

$$P[\eta(z)] = \exp \left[-\frac{1}{2h} \int_0^z \eta^2(z) dz \right]. \quad (15)$$

Using this distribution, the propagator K , which is connected between two end points of pseudo-spin, is given by the functional integral:

$$K[\mathbf{S}(z)|\mathbf{S}(0)] = \int \prod_z \delta \left[\frac{d\mathbf{S}}{dz} + \mathbf{A}(\mathbf{S}(z)) - \eta(z) \right] \times \exp \left[-\int \frac{\eta^2(z)}{2h} dz \right] \mathcal{D}[\mathbf{S}] J(\mathbf{S}) \mathcal{D}[\eta(z)] \quad (16)$$

with δ being the Dirac delta-functional, and $J(\mathbf{S})$ is the functional Jacobian. By carrying out the Gaussian functional integral with respect to $\eta(z)$, one obtains

$$K[\mathbf{S}(z)|\mathbf{S}(0)] = \int \exp \left[-\frac{1}{2h} \int_0^z \left(\frac{d\mathbf{S}}{dz} + \mathbf{A}(\mathbf{S}) \right)^2 dz \right] J(\mathbf{S}) \mathcal{D}[\mathbf{S}]. \quad (17)$$

where $J(\mathbf{S})$ is given as

$$J(\mathbf{S}) = \exp \left[\int_0^z M dz \right] \quad (18)$$

and we see that M is determined as $M = -\frac{1}{2} \frac{\partial}{\partial \mathbf{S}} \cdot \mathbf{A}$ [19].

5. THE FOKKER-PLANCK EQUATION

The derivation of the FP equation is carried out most directly by using the above path integral. For this purpose, we use the “imaginary time trick”, that is, we define $\tau = iz$, so introduce the “wave function” $\Psi(\mathbf{S}, \tau)$, then we have the integral equation:

$$\Psi(\mathbf{S}, \tau) = \int K[\mathbf{S}(\tau)|\mathbf{S}(0)]\Psi(\mathbf{S}, 0)d\mathbf{S}(0). \quad (19)$$

Following the standard procedure of Feynman path integral, we obtain the Schrödinger equation:

$$ih \frac{\partial \Psi}{\partial \tau} = -\frac{1}{2} (\mathbf{p} - i\mathbf{A})^2 \Psi + W\Psi \quad (20)$$

where

$$\mathbf{p} = -ih \frac{\partial}{\partial \mathbf{S}}, \quad W = \frac{\mathbf{A}^2}{2} + Mh \quad (21)$$

By using Eq. (21), replacing the imaginary time τ with the original real coordinate z , namely $\tau \rightarrow iz$, and getting “wave function” Ψ back to the original probability distribution P . Thus, we can obtain the FP equation:

$$\frac{\partial P}{\partial z} = \frac{h}{2} \left(\frac{\partial}{\partial \mathbf{S}} \right)^2 P + \frac{\partial}{\partial \mathbf{S}} \cdot (\mathbf{A}P) \quad (22)$$

The detailed analyses for this FP equation will be given near-future.

6. SUMMARY

We have developed the theory of evolution of the polarization inherent in the non-linear birefringent media. This can be realized as the dynamics of the Stokes parameters. The equation of the Stokes parameters is reduced to the Langevin equation. The Langevin equation is naturally converted to the FP equation by using the functional integral based on the Gaussian white noise for the random fluctuations inherent in the birefringent media.

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