# Geometry and Its Physical Meaning

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**Abstract**— Until the end of 1800s, the graphical aspect of geometry used to be compared to the external world, called "physical space", and justified on the basis of mechanics. As a consequence of the foundational crisis of mathematics, geometry switched to axiomatic programs and disengaged from the physical intuition of space. Its logical structure was formalized and further developed, neglecting geometrical constructions. Endorser of formalist programs, physics evolved toward more and more sophisticated interpretations of the world, also supported by advancements of surveying and measurement techniques. Physical laws are regarded as interpretative algorithms tied to recurring process patterns, no longer as interpretations of a code that Nature is written in the language of. What got lost during that transition, in our opinion, is the ability to avail of both experience and deduction in geometrical models of field theories, so as to provide a logical foundation for interpreting the extent in a physically meaningful way. In particular, (1) irrespectively of whatever code, the optical channel does convey a huge amount of the data, which our experience consists of, and (2) geometry allows to model non-optical electromagnetic signals consistently with the optical ones, so as to analyze them uniformly. In this paper, we stem from models entailed by geometrical optics and discuss the relationships among them, imaging based on projective geometry, and mathematical analysis.

## 1. INTRODUCTION

Euclid's contribution to geometry as we know it consisted in setting logical reasoning apart from cosmological conceptions, sensory perceptions, and empirically gathered methods to assess properties of physical objects.[1] His distinction between postulates and axioms, problems and theorems[2] suggest that he regarded the drawings constructed to illustrate theorems as proofs of existence [1].[3] Newton diverged from that mindset. In his Principia, he combines cosmology and Euclidean geometry, originating the concept of a space devoid of matter. In the early 1900s, as a consequence of the uptake of non-Euclidean geometry, the whole discipline, including Euclidean, non-Euclidean, affine and projective geometries, was restated on an axiomatic basis. Today implicit designations of terms such as point and straight line,[4] given through axioms, supersede explicit definitions of graphical elements endowed with geometrical meaning. Logical principles of geometrical thinking introduce abstraction via isomorphisms, such as that between real numbers and a straight line.

Based on isomorphisms, analytical representations of axiomatic structures bear a semantical meaning different from that of traditional geometrical constructions. For an idea of the meaning of constructions, consider how Mach verified the parallelogram of forces by probing Galileo's and Stevin's experiences on inclined planes [2]. His graphical construction does neither suggest what inertia is, nor prove laws of forces or mathematical parallelogram identity. However, it shows how Varignon's construction applies to Galileo's and Stevin's mechanics. On the other hand, to say that vectors *realize* linear algebra is a statement which does not depend on any application or graphical symbol.[5] Suppression of geometrical constructions has eventually implied waiver of real world modeling.

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[1]We mean geometry as a deductive process, not as Platonic ideas ($\iota\delta\epsilon\alpha$ = archetypal image) like Proclus does.

[2]It is difficult, today, to estimate what Euclid meant by his porisms. According to pre-Socratic philosophers, through whom the idea of Cosmos came, it made no sense to attribute a shape to a vacuum, as shapes belonged to the four elements, earth, air, water, and fire. Celestial maps of the time display the vault of sky held up by the ocean around the flat Earth. Later, ∼350 BC, at Plato's and Aristotle's time, Earth shape became spherical, and the wheels, which used to trail celestial bodies around the Earth in the earlier, groundbreaking model of Anaximander-Thales' disciple-, were replaced by armillary spheres. The cutting-edge cosmological problem was "Why doesn't the Earth fall down?" By contrast, Euclid's Elements, Book XI, which dates back to that time, introduces solids without referring to physical space. Indeed, it neither defines a "space".

[3]While a theorem's statement establishes a universal quantification (∀), a geometrical construction gives an existential one (∃).

[4]In his sixth problem, Hilbert proposed to treat physics substituting the basic objects of the axiomatically formulated geometry with a different system of objects put in a one-to-one relationship with it.

[5]Valid mathematical statements don't imply the existence of the entities that they refer to, except where explicitly noted. Furthermore, mathematical existence does not imply any corresponding physical object.

## 2. CONSTRUCTIVE GEOMETRICAL APPROACH TO OPTICS

The geometric framework of electromagnetism results mainly from two approaches. (1) The electrodynamic approach which, thanks to Lagrange's analytical mechanics formulation, evolved into the unified theories of space-time geometry with quantum theoretical principles. (2) The graphical ray-optics approach, which is dual to Huygens' wave fronts. In the latter approach, constructions are borrowed from the so-called *mechanical geometry*, where optical instruments play the role of ray tracer, a kind of geometric construction tool. In the rest of this section we give a critical account of the way approach (2) describes magnification $M$ as obtained by optical systems without taking defects and apertures into account. The fundamental assumption of geometrical optics is that the angular magnification is the same as the linear one, so that an image is similar to the corresponding object.[6] A light ray emitted by a source point $P_1$ is plotted as a straight line having a slope $\alpha_0$ with respect to the optical axis $x$.[7] The optical instrument provides for tracing the ray to the final image point, say $P_{N+1}$. Analytical geometry provides a method to convert the resulting construction to an analytical expression, as we do next [3].

Let's denote by $S_1, \ldots, S_N$ a coaxial array of thin lenses; by $d_j$ the distances $(S_j, S_{j+1})$; by $P_1$ the object point, and by $P_2, \ldots, P_{N+1}$ its images; by $s_i$ the distances between any two of those, so that $s_0$ is the distance $(S_1, P_1)$, $s_1$ the distance $(S_1, P_2)$ and so forth.[8] As images are formed, the system traces open, monotone polygonal chains, whereby the image point after $S_j$ works as an object point for for the next lens. In the one-simple-lens case, the correspondence between graphics and experiment can easily be verified. A normal congruence of traced rays can be visualized as a beam, e.g., by spreading around some smoke. That beam may suggest to consider a ray as a light trajectory. The lens first $F$ and second $F'$ focal points can be identified that way. The focal length is $f = FS_1 = S_1F'$. Let's call $n$ the air-glass refractive index, and $1/r$, $1/r'$ the curvatures, then the power is defined as: $p := \frac{1}{s_1} - \frac{1}{s_0} = (n-1)(\frac{1}{r} - \frac{1}{r'}) = \frac{1}{f}$.

For a system composed of $N$ lenses, let's compute the abscissa of the image point $P_{N+1}$ formed at a distance $s_{2N-1}$ behind the last lens, $S_N$. Starting from the identities $f_1(1/s_1 - 1/s_0) = f_2(1/s_3 - 1/s_2) = \ldots = 1$ valid for each lens, a system of $N$ equations is build up solving for even $s_i$: $s_0 = \frac{-f_1}{1-f_1/s_1} = \frac{-f_1}{1-\frac{f_1}{d_1+s_2}}, s_2 = \frac{-f_2}{1-\frac{f_2}{d_2+s_4}}, \ldots, s_{2N-2} = \frac{-f_N}{1-\frac{f_N}{s_{2N-1}}}$. By substitution, we get the descending continued fraction: $s_{2N-1} = \frac{f_N}{1-}\frac{f_N}{d_{N-1}-}\frac{f_{N-1}}{1-}\frac{f_{N-1}}{d_{N-2}-}\frac{f_{N-2}}{1-}\frac{f_{N-2}}{d_{N-3}-}\cdots\frac{f_2}{1-}\frac{f_2}{d_1-}\frac{f_1}{1+f_1/s_0}$. That expression consists of simple terms, and resembles an *amplitude function* of ladder networks [4, 5].[9] In geometrical optics it is derived from the physical laws that light-related phenomena obey to.

If the object has an extent in a plane $x = const$, and diffuses light toward the glass-work as a whole, the size $h'$ of the image is magnified relative to the size $h$ of the object. In order to determine the transverse magnification $M = h'/h$ choose a point $P_1$ of the object and trace a ray from there toward $S_1$, such that the ray meets its vertex with height $y_0 = 0$ and slope $y_{S_1} = s_0\alpha_0$. As the lens has a power, the ray leaves with an increased slope $p_1 y_{S_1} = \alpha_1 - \alpha_0$, resulting in a ratio of $\alpha_1/\alpha_0 = 1 + p_1 s_0$. The outgoing ray has equation $y = \alpha_1(x - x_{S_2}) + \alpha_1 d_1$. The ordinate undergoes a further increment $y_{S_2} - y_{S_1} = \alpha_1 d_1$ along the subsequent segment $d_1$ between $S_1$ and $S_2$.[10] Repeating the procedure for each lens, one obtains a system of equations with parameters $\alpha_j$ and $y_{S_{j+1}}$:[11]

[6]Euclidean geometry defines similarity as an admissible transformation. In optics, similarity applies to objects perpendicular to the optical axis. It is confined to paraxial approximation and thin lenses, not dioptric systems such as eyes.

[7]The so called divergence angle $\alpha_0$ (with sign) of the ray is related to the incidence angle $i$ in Snell's sine law by $i = \theta - \alpha_0$. $\pm\theta$ is the angle between $x$ and the normal to the refractive surface meeting that light ray. For consistence $-\theta$ is used for a convergent lens.

[8]Here, we follow the Cartesian sign convention. The $x$ axis is oriented toward increasing indexes; $S_1$ is in $x = 0$. Intermediate distances with odd subscripts are oriented as $x$, while $s_{2j} = s_{2j-1} - d_{2j-1}$ have the opposite orientation.

[9]Lenses form more than one image, if all reflections and refractions are taken into account. Here we are dealing with the geometric refraction of a point object located on the $x$-axis, disregarding diffraction and aberrations, frequency dispersion included, as well as the diffraction effects of apertures. The system's ray-transfer matrix reduces to a quotient of polynomials like a transfer function of a passive linear filter. When Cauer's mathematical development is carried over to optics, the dichotomy between (diffraction) frequency- and space-domain can be abandoned. In fact, Stieltjes showed how to express equivalently any holomorphic function either by a possibly infinite continued fraction, or by a definite integral. Hence, the frequently encountered linear functional transform $\int_0^\infty \psi(u)e^{zg(u)}\,du$ of an analytical function $\phi(z)$, $z \in \mathbb{C}$ can be represented as a continued fraction. Continued fractions converge in instances where the corresponding series do not. For an alternative treatment of filters, see, e.g., Bloomfield.

[10]Lens $S_j$ contributes $\Delta_j\alpha = \alpha_j - \alpha_{j-1} = p_j y_{S_j}$ and $\Delta_{S_j} y = d_{j-1}\alpha_{j-1}$. If the $j$-th segment is inside a medium having refractive index $n_j$, one replaces $d_j$ with $t_j = d_j/n_j$.

[11]Given $y_{S_1} = s_0\alpha_0$, The tridiagonal matrix is formed from the rays' parameters: $(\alpha_1 = \alpha_0 + p_1 y_{S_1}, y_{S_2} = y_{S_1} + d_1\alpha_1)$,

$$
\begin{array}{cccccccc}
\alpha_1 & & & & & = & \alpha_0 & +p_1 y_{S_1} \\
-d_1\alpha_1 & +y_{S_2} & & & & = & & y_{S_1} \\
-\alpha_1 & -p_2 y_{S_2} & +\alpha_2 & & & = & 0 & \\
0 & -y_{S_2} & -d_2\alpha_2 & +y_{S_3} & & = & 0 & \\
0 & 0 & -\alpha_2 & -p_3 y_{S_3} & +\alpha_3 & = & 0 & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}
$$

The parameters ($\alpha_N$, $y_N$) belong to the ray which leaves $S_N$ to form the final image. Expressing them as a function of the input ray parameters ($\alpha_0$, 0), by substitution, gives a continued fraction like the one obtained above for $s_{2N-1}$. In fact, the total power $p$ could have been obtained from the above continued fraction by solving it for $f_i$. In Gauss' description, the magnification is given by the determinant of the coefficient matrix as $h/h' = \alpha_m/\alpha_0 = ps_0 + \partial p/\partial p_1$, where the latter expression is the cofactor expansion, in which the coefficient of the second term is the formal derivative with respect to $p_1$ of the first one.[12]

Since both abscissa and magnification are built in an Euclidean space, and each ray used for the pointwise construction can be visualized, the image made up of a point from each ray should be similarly traceable in the same space, as if it got off the object as an empty form and proceeded through the glass-work [6][13]. In fact, a splotch of light can be visualized on a cloud of smoke or a screen, but no propagation of an ethereal, object-like image is experimentally substantiated. The "extent", a term used to mean *that* property of the external world, is not modeled. Hence, interest turned from the homogeneous and isotropic space of Euclidean geometry toward the algebraic program of Lagrangian mechanics and Gauss' differential geometry.[14] Consequently, algebraic geometry applied to optics turned to formulating just the theoretical framework. Concepts such as light rays remain as relics of the previous theories and are only used to understand what we're talking about.

## 3. GEOMETRY AS A MODEL OF THE EXTENT

Geometry owes its name to the fact that, since the times of King Scorpion, Egyptians used tools equivalent to straightedges and compasses to parcel out the terrain after Nile floods. As Greek symbols hardly lent themselves to numerical computing, in Books V and VI Euclid developed compass-and-straightedge constructions to compute ratios between quantities.[15] Today, we associate continuous numerical variables to geometric objects — in an analytic, algebraic, or other context — considering metric as an integral part of geometry. Instead, geometrical shapes model plots of land as long as the same tools are used, at least theoretically, in both cases.[16]

*Mechanical geometry* (kinematics), which we touched upon about geometrical optics constructions, is a geometry different from Euclid's in that it originated for squaring the circle, and related Delian problems which could not be solved using only straightedge and collapsible compass [7]. It deploys gears motion to trace curves so that their measures depend on an algorithm of motion rather than on deductive reasoning and Euclid's construction tools.[17] When the Helmholtz-Lie problem of generating space by means of motion was put forward, the transformations performed

($\alpha_2 = \alpha_1 + p_2 y_{S_2}$, $y_{S_3} = y_{S_2} + \alpha_2 d_2$), ..., whose determinant of the coefficients is a continuant polynomial.

[12] The formal derivative is obtained by removing from $p$ the row and column which contain coefficient $p_1$. The magnification $M = h'/h$ of the system, expressed as a function of the powers of the lenses and their separation, varies inversely as the distance $s_0$ of the object. This result makes use of the Lagrange invariant $n'h'\alpha' = nh\alpha$.

[13] Lucretius (deceased in 55 BC) tells that the image detaches itself from a body like a pellicle. He would not have admitted that a pinhole camera yields a clear image. Since then, there have been huge experimental and mathematical developments. However, the geometric theory of aberrations still recurs to the Gaussian image to determine the light intensity distribution. The wave-fronts proceeding from each element of a body orthogonally to the beams — whose departure from the spherical shape determines aberrations — quite resemble Lucretian pellicles.

[14] Originally, Hamilton derived the laws of reflection and refraction of light rays from the principle of least action given in Jacobi's form $A = \int_1^2 \sqrt{(E_{TOT} - V)\sum_{i,j} a_{ij}\, dq_i\, dq_j}$, where $V$ is a suitable potential energy, $q_k$ $(k = 1, \dots, n)$ are generalized coordinates, and the initial and final configurations have subscripts 1 and 2 respectively. In a differential geometric interpretation of the principle, $q_k$ are intrinsic coordinates on a Riemann surface, and the differential element of a geodesic on it is the infinitesimal path length of a light ray $ds^2 = (E_{TOT} - V)\sum_{i,j} a_{ij}\, dq_i\, dq_j$. By substituting $ds \to nds$, where $n$ is the refractive index of an isotropic homogeneous medium, a surface orthogonal to a rays congruence can be interpreted as Hamilton's characteristic function *applied to optics* on that bidimensional manifold having said metric. If one considers wave fronts instead of rays, $A$ must be interpreted as time (Fermat's principle of stationary action). Expressing the action of $N$ point particles in that framework, Lipschitz contributed a dynamical foundation to Hamilton's formulation. The geometrical representation in the configuration space is due to him.

[15] Euclid used to draw upon geometry to treat problems of algebraic or computational nature (in modern conception).

[16] A measurement of length presupposes a measuring rod and a surface plate. Along a rough path, a straight line "drawn" by theodolite likely yields a different length value then that drawn using a tape.

[17] Roulettes, the curves generated by a point on a rolling gear, have been Newton's starting point.

by optical instruments were written as differentials for light propagation, and were interpreted according to intrinsic geometry. It isn't allowed, though, to identify the means of representation — the light rays — with the represented object — the image.

Consistently with the Erlangen Program, Klein's model presents Maxwell's field equations as an invariant under geometric transformations of *projective space*. Once the aether medium has been relinquished, it accounts for the input-output transformations typical of electromagnetic signals.[18] Projective geometry is a theoretical framework consisting of deductive reasoning and a construction tool — the straightedge. Graphically, it gives views of a space constructible by straightedge, whereas geometrical optics generates its space by ray tracing. In order to allow consistent interpretations, the geometric framework of electromagnetism should be the same for all signals in every frequency range.[19] In the rest of this section, we try to justify the choice of a projective-space model.

According to Maxwell, most of the relations between an object and its image formed by an optical instrument may be deduced from the purely geometrical theory of homographic figures [8, 9].[20] Abscissa $x$ of a source point is transformed into the corresponding sink point abscissa $t(x) = \frac{\alpha x+\beta}{\gamma x+\delta} = \frac{\alpha}{\gamma} - \frac{D/\gamma^2}{\frac{\delta}{\gamma}+x}$, where $D = \alpha\delta - \beta\gamma$. Parameters $\alpha$, $\beta$, $\gamma$, and $\delta$ are related to the instrument, which is considered a black box, and are independent of the source point. The projective transformation — a generalization of perspective — is obtained, as previously, as the sum of simple terms. For example, if the $j$-th term has the form $t_j(x) = \frac{a_j}{b_j+x}$, by letting $T_0(x) = t_0(x) = s_0 + x$ and $T_j(x) = T_0(t_1 t_2 \dots t_j(x))$, we have the successive convergents $T_1(x) = T_0(t_1(x)) = s_0 + t_1(x) = s_0 + \frac{a_1}{b_1+x}$, $T_2(x) = T_1(t_2(x)) = s_0 + \frac{a_1}{b_1+\frac{a_2}{b_2+x}}$, and so forth.

The expression for the transformation of the object-point (source) into the image-point (sink) is similar to those obtained previously [10].[21] The projective transformation group is a proper superset of point-to-point transformations (homographies), so the transformations could be extended from systems composed of $N$ lenses to general linear receivers, and projective *models* of the extent could help interpreting small signals in any frequency range [11].[22]

Ancient geometry had a secular propensity for visual models. Euclid is considered the (probable) author of a pamphlet where the Greek knowledge on vision was summarized in the form of notions and theorems on perspective. It was the starting point of relevant arts and crafts upon its rediscovery at the dawn of Renaissance. Painters conceived central projection as a means to render the illusion of a spatial disposition of objects to an observer looking at the painting from a predetermined position.[23] Sculptural reliefs and architectonic structures were also studied. Modern science of vision, by contrast, concedes to ethereal images only a perceptual space internal to the mind, and no extent in the outer world. It arouses research interest rather about psychological facets of optical impressions than geometric models [12]. Accordingly, the topology of the visual space is intermingled with statements on physiological structures and functions. Geometry of position exists, along with other legacy achievements. Some projective constructions were investigated as to how they match their Euclidean counterparts, but research purely on projective constructions was never

[18]Plücker's line complex — the null system — belongs to projective geometry. The transformations of his complex were connected to the system of partial differential equations that summarizes the behavior of electromagnetic fields in a vacuum. As those fields are defined over the whole space, the logical framework provided by a geometric model is by far more important for their representations than it is for point particle dynamics.

[19]The oneness of the geometric framework sounds obvious. Projective geometry models the extent if it is agreed upon that visual images are phenomena, and as such pertain to the *external* world. However, images viewed in the light frequency range are not to be confused with projected silhouettes. In other frequency ranges, signals received are often reconstructed as if they were slide shows. In our opinion, rendering them in a way homogeneous with optical images would allow to compare the data on the same logical basis. Comparing contours would be a different model.

[20]In "Draft on the methods of geometrical optics. Appendix: Homographic figures and projective geometry", 1874.

[21]The transform $T$ results from successive iterations. It can be expressed by a definite integral $T(\phi) = \int_a^b \phi(u)K(u)du$, where the part $K$ describing the performance of the receiver is a well behaved function. See also note 9. Volterra introduced his *functions of a line* $T(\phi)$ as classes of scalar valued functions which depend on the shape of a curve, construed as the limit of functions defined on polygons as the number of vertices approaches infinity, pioneering functional calculus. His work can be generalized by considering correspondences between elements of the same space which contain additional parameters $\phi(x, y, z) \xrightarrow{T} \psi(x, y, z)$, thereby expressing the received signal as a transformation of the space onto itself. For a different approach, Fredholm's definition of the integral equation is rather oriented toward the solution of differential equations, and generalizes linear algebraic systems by providing for infinitely many variables. Both methods can use continued fractions to obtain approximated solutions of integral equations in the framework of Helmholtz-von Neumann spectral theory.

[22]An imaging problem is how to represent *signals* geometrically when the frequency range is not visually perceivable. In fact, human visual experience is integrated with knowledge originated from the other senses, while non-visible electromagnetic signals, however well rendered, can hardly be endowed with an equivalent evocative power.

[23]Photographic techniques meet the same demand as painting.

undertaken. In an attempt to elicit technical interest, let's mention two facts about images. One, a subject in daylight forms a sharp image whether it projects on a flat screen through a pinhole, or it appears inside a crystal sphere as a 3D image.[24] Two, Fourier optics lacks a geometric model for the observed Fraunhofer diffraction pattern, although Abbe showed that such patterns are images at finite of subjects at the optical system's infinity, or vice-versa.

## 4. AT THE ROOTS OF PROJECTIVE GEOMETRY

The roots of displaying forms in projective geometry are twofold. One is the theory of perspective, a theorization of artistic painting. It links back to Euclid's and Apollonius' methods, but, as a theory, it does not even cover all of constructions actually used by architects.[25] The other root — geometry of position — is traced back to Desargues and Pascal about 1600. They conceived constructions independent of any projection point; that is, figures with no metric properties. However, a logical-deductive system to back their constructions was missing. Toward the end of 1700s, the critical review of the parallel postulate and the mathematical foundations research it triggered determined the convergence of graphical perspective with Desargues' and Pascal's constructions, which resulted in a new *synthetic geometry* approach, entirely different from Euclid's one. Finally, F. Klein proved that the projective logical-deductive system is self-standing, as it can be stated independently of the Euclidean one. Unfortunately, since its foundation, that new geometry was progressively derailed toward *analytic geometry*.[26] The resulting formalism is homogeneous across all geometries, but does not recognize any more the differences between a system that only avails of a straightedge and the one formalized by Euclid. Since projective transformations can be handled with algebraic methods, their group structure can be interpreted in a manifold. However projective configurations can be contrasted against Euclidean figures.[27] Having dealt with projective models of non-Euclidean geometries, Klein concluded that a non-Euclidean surface is endowed with a curvature only if it is immersed in an Euclidean space; the geodesics of the same surface are straight lines in projective geometry.[28] He suggested that physical models can be built upon projective geometry just like mathematical ones. That suggestion anticipated image reconstruction techniques, as well as all the problems related to sampling, storage, and recovery of information. Nowadays, backed up by considerable experience in treating and rendering encoded signals, the idea of modeling electromagnetic signals directly upon projective geometry could be taken up again. Geometric modeling can supersede — and surpass — nomograms, for example the Smith chart, bringing forth the power of a full blown hypothetico-deductive system in stead of specifically tailored graphical aids.

## 5. CONCLUSIONS

Geometry attained mathematical rigor by neglecting the relationship between modeling of the display and geometrical constructions, both of which hence lack mathematical support. Our analysis suggests that, restoring that relationship, projective geometry can lend itself to modeling the extent in electromagnetism, in linear approximation. "Linear" in both ways: (1) geometrical, as geometric space is homogeneous and isotropic, and (2) electromagnetic, according to the so-called "small signal" approximation of fields. Since the axiomatization of mathematics, little progress was made which would be useful to physics in this respect. Notwithstanding today's technical achievements, modeling based on projective geometry has remained in a somewhat rudimentary state. In our opinion, modeling of received signals could provide a common ground for information theories applied to telecommunications and information-based complexity theories. In fact, the huge advancements in electromagnetics and computer sciences make it also possible to gradually broaden the frequency ranges that we can interpret visually, under varying illumination conditions.

[24] The image we refer to appears right inside the sphere, and is not the only one being formed. The imaged subject is always the whole world, not just the detail of interest.

[25] Panoramic drawings use multiple projection points to correct perspective, so as to take into account binocular visual effects at a slant. We underline this detail because, when the geometric construction itself is the subject under investigation, the ability to represent projective spaces in homogeneous coordinates deserves (re)consideration.

[26] Hilbert solved definitively the consistency problem by reducing all geometries to a unique Cartesian geometry.

[27] Projective constructions differ from Euclidean ones in how elements at infinity are conceived and in the duality principle. As for Seidel aberrations, observed in optics, projective transformations of a straight line range all conics. In fact, any conic can be construed using only a straightedge, according to Steiner.

[28] Projective geometry provides models of non-Euclidean geometries. They are not the only models. Poincaré represented hyperbolic geometry on a disc in the complex plane. In his model, conformal but not projective, the unit disc in $\mathbb{C}$ results from the mapping of the Riemannian surface generated by the group of automorphic functions by tiling of their fundamental domains.

**REFERENCES**

1. Rodin, A., "On constructive axiomatic method," *Specific Sciences — Mathematics*, arXiv: 1408.3591v2, 2014.
2. Mach, E., *The Science of Mechanics*, The Open Court Publishing Co., Chicago, 1919.
3. Herman, R. A., *A Treatise on Geometrical Optics*, University Press, Cambridge, 1900.
4. Cauer, W., *Synthesis of Linear Communication Networks*, McGraw-Hill, New York, 1958.
5. Bloomfield, P., *Fourier Analysis of Time Series. An Introduction*, Wiley-Interscience, New York, 2000.
6. Pfeil, A., F. Wyrowsky, A. Drauschke, and H. Aagendahl, "Analysis of optical elements with the local plane-interface approximation," *Appl. Opt.*, Vol. 39, No. 19, 3304–3313, 2000.
7. Braunmühl, A., "Historische Studien über die organische Erzeugung ebener Curven von den ältesten Zeiten bis zum 18. Jahrhundert," *Katalog Mathematischer und Mathematisch-physikalischer Modelle, Apparate und Instrumente*, C. Wolf & Sohn, München, 1892.
8. Harman, P. M., *The Natural Philosophy of James Clerk Maxwell*, Cambridge University Press, Cambridge, 2001.
9. Whittaker, E. T., *The Theory of Optical Instruments*, Cambridge University Press, Cambridge, 1907.
10. Fantappiè, L., "Überblick über die theorie der analytischen funktionale und ihre Anwendungen," *DMV-Jahresbericht*, Vol. 43, 1–25, 1934.
11. Biederman, I., "Human image understanding. Recent research and theory," *Comput. Vision Graph.*, Vol. 30, No. 1, 29–73, 1985.
12. Ronchi, V., "Classical optics is a mathematical science," *Arch. Hist. Exact Sci.*, Vol. 1, No. 2, 160–171, 1960.