Homogenizations of Micropolar Elastic Metamaterial Using Field Averaging

C. N. Weng and T. Chen
Department of Civil Engineering, National Cheng Kung University, Tainan, Taiwan

Abstract— In this paper, we introduce an approximation in homogenization of a three dimensional elastic metamaterials by a homogeneous equivalent micropolar media whose effective parameters are determined by average theory. Formulations of averaged physical fields over the representative volume element (RVE) together with dispersion relations of plane wave propagation are presented in the text.

1. INTRODUCTION

Micropolar elasticity, also called Cosserat elasticity [1, 2], is a generalized continuum theory. In contrast to classical continuum, each material point in a micropolar medium has six degree of freedom, which often be used to describe solid materials with intrinsic microstructure effects like bone [3], porous material [4–6], brick structure [7], granular materials [8, 9] and many others. Recently, inspired by electromagnetic metamaterials [10], the research of elastic metamaterials have attracted considerable interest due to potential applications in manipulating the propagation of elastic waves [11–17]. Among them, a few reported works focus on the design of chiral microstructures which can enhance rotational resonance and thus achieve double negative effective parameters [14, 15]. The rotational motion has been observed as a fundamental mode of elastic wave propagation in phononic crystals [18–20]. However, less attention has been paid to the issues of effective micropolar constants in model analysis, the conventional effective medium theories for elastodynamics put more emphasis on Cauchy solids [21, 22] or Willis constitutive relation [23, 24]. In this paper, we attempt to propose an approximation in homogenization of heterogeneous elastic metamaterials by a homogeneous equivalent micropolar media whose effective parameters are determined by average theory. It is convince that micropolar theory is appropriate to analyze the metamaterial model than traditional continuum theory due to the complex local resonance. We illustrate a simple model with cuboid periodically arranged structures to demonstrate the calculation processes of effective parameters and the dispersion relation of plane wave propagation. Comparing to other literatures which was devoted to the modeling of two dimensional micropolar medium [7, 25–27]. the presented methodology can be applied to homogenize the three dimensional periodic structures with unit cells having micropolar effects.

2. BASIC EQUATIONS OF MICROPOLAR ELASTICITY

The key difference between a micro-polar and Cauchy medium is that each material point can not only translate but rotate independently. Additional degrees of freedom, micro-rotations ϕ_i, are introduced in linear theory of micropolar elasticity. The kinematics can be expressed as

$$\varepsilon_{ij} = u_{j,i} - e_{ijk}\phi_k, \quad \kappa_{ij} = \phi_{j,i}, \tag{1}$$

where ε_{ij}, κ_{ij} and e_{ijk} are the strain, curvature and permutation tensors, respectively. The equations of motion are

$$\sigma_{ji,j} = \rho\ddot{u}_i, \quad \mu_{ji,j} + e_{ijk}\sigma_{jk} = J\ddot{\phi}_i, \tag{2}$$

where σ_{ij} and μ_{ij} represent stress and couple-stress tensors, respectively. ρ and J are the density and rotational inertia, respectively. For a centro-symmetric micropolar medium, the constitutive relations are given as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad \mu_{ij} = D_{ijkl}\kappa_{kl}, \tag{3}$$

in which C_{ijkl} and D_{ijkl} are elastic tensors. It should be note that the tensor fields, ε_{ij}, κ_{ij}, σ_{ij} and μ_{ij}, are usually asymmetric in micropolar continuums, thus the constitutive moduli tensors, C_{ijkl} and D_{ijkl}, no longer possess minor symmetries but fulfill the major symmetric property. It is mention that the number of independent components of C_{ijkl} (or D_{ijkl}) is 45 for general anisotropic micropolar solids and reduces to 3 for isotropic cases.

3. HOMOGENIZATION OF MICROPOLAR MEDIUM

In this section we present a numerical homogenization procedure that relates the microscopic local fields like displacement and stresses to macroscopic quantities like rotation and couple stress. Apart from traditional models, we assume that the homogenized model behaves as a micropolar solid at macroscopic scale while each constituent remains a classical Cauchy continuum on the microscale. Consider a 3D metamaterial structure sketched in Fig. 1 whose unit cell is a rectangular cuboid with sides d_1, d_2 and d_3. Due to the translational symmetry, the Bloch's theorem and periodic boundary conditions are applied in our analysis. To describe the overall behavior of the model, we utilize the concept of field homogenization. The averaging process of physical fields are applied over the unit cell as

$$\bar{u}_j = \frac{1}{V}\int_V u_j dv, \quad \bar{\phi}_j = \frac{1}{V}\int_V \phi_j dv, \tag{4}$$

$$\bar{\varepsilon}_{ij} = \frac{1}{V}\int_V \varepsilon_{ij}(\vec{x}) dv, \quad \bar{\kappa}_{ij} = \frac{1}{V}\int_V \kappa_{ij}(\vec{x}) dv, \tag{5}$$

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_V \sigma_{ij}(\vec{x}) dv, \quad \bar{\mu}_{ij} = \frac{1}{V}\int_V \mu_{ij}(\vec{x}) dv, \tag{6}$$

where V is the volume of the unit cell. The over bar denotes the averaged quantities. We mention that the integration of microrotations and couple stresses cannot be directly computed due to our original assumptions so that alternative approximation is demanded. In this model, the average microrotations can be approached by integrating boundary displacements as

$$\begin{aligned}
\bar{\phi}_1 = \frac{1}{8V}\int_{-\frac{d_1}{2}}^{\frac{d_1}{2}} \Bigg(&\int_{-\frac{d_3}{2}}^{\frac{d_3}{2}} u_3\big|_{x_2=\frac{d_2}{2}} dx_3 - \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} u_2\big|_{x_3=\frac{d_3}{2}} dx_2 \\
&- \int_{-\frac{d_3}{2}}^{\frac{d_3}{2}} u_3\big|_{x_2=-\frac{d_2}{2}} dx_3 + \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} u_2\big|_{x_3=-\frac{d_3}{2}} dx_2 \Bigg) dx_1, \\
\bar{\phi}_2 = \frac{1}{8V}\int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} \Bigg(&\int_{-\frac{d_1}{2}}^{\frac{d_1}{2}} u_1\big|_{x_3=\frac{d_3}{2}} dx_1 - \int_{-\frac{d_3}{2}}^{\frac{d_3}{2}} u_3\big|_{x_1=\frac{d_1}{2}} dx_3 \\
&- \int_{-\frac{d_1}{2}}^{\frac{d_1}{2}} u_1\big|_{x_3=-\frac{d_3}{2}} dx_1 + \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} u_3\big|_{x_1=-\frac{d_1}{2}} dx_3 \Bigg) dx_2, \\
\bar{\phi}_3 = \frac{1}{8V}\int_{-\frac{d_3}{2}}^{\frac{d_3}{2}} \Bigg(&\int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} u_2\big|_{x_1=\frac{d_1}{2}} dx_2 - \int_{-\frac{d_1}{2}}^{\frac{d_1}{2}} u_1\big|_{x_2=\frac{d_1}{2}} dx_1 \\
&- \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} u_2\big|_{x_1=-\frac{d_1}{2}} dx_2 + \int_{-\frac{d_1}{2}}^{\frac{d_1}{2}} u_1\big|_{x_2=-\frac{d_1}{2}} dx_1 \Bigg) dx_3,
\end{aligned} \tag{7}$$

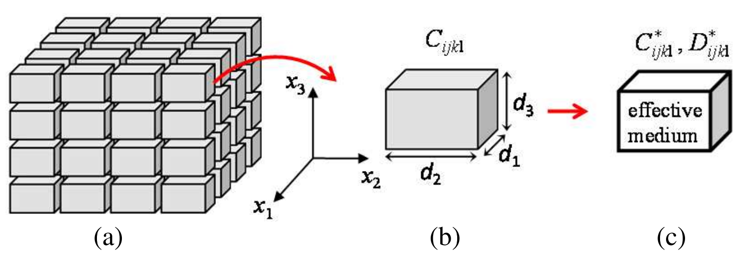

Figure 1: Illustrations of (a) 3D elastic metamaterial structure, (b) a unit cell and (c) effective model of the metamaterial. The unit cell is a traditional Cauchy medium, whereas the effective model is assumed to be a micropolar medium.

and thus Eq. (5) can be rewritten in the following form

$$\bar{\varepsilon}_{ij} = \frac{1}{V}\int_V \frac{\partial u_j}{\partial x_i} - e_{ijk}\phi_k dv = \frac{1}{V}\oint_S u_j n_i ds - e_{ijk}\frac{1}{V}\int_V \phi_k dv = \frac{1}{V}\oint_S u_j n_i ds - e_{ijk}\bar{\phi}_k, \tag{8}$$

$$\bar{\kappa}_{ij} = \frac{1}{V}\int_V \frac{\partial \phi_j}{\partial x_i} dv, = \frac{1}{V}\oiint_S \phi_j n_i ds = \frac{2\bar{\phi}_j}{d_i}. \tag{9}$$

Applying divergence theorem to Eq. (6) and combining Eq. (2) leads to

$$\begin{aligned}\bar{\sigma}_{ij} &= \frac{1}{V}\int_V (\sigma_{kj}x_i)_{,k} - \rho\ddot{u}_j x_i dv = \frac{1}{V}\left(\oiint_s \sigma_{kj}n_k x_i ds - \int_V \rho\ddot{u}_j x_i dv\right)\\ &= \frac{1}{V}\left(d_I\int_S \sigma_{ij} ds_I - \int_V \rho\ddot{u}_j x_i dv\right),\end{aligned} \tag{10}$$

$$\begin{aligned}\bar{\mu}_{ij} &= \frac{1}{V}\int_V (\mu_{kj}x_i)_{,k} + \in_{jkl}\sigma_{kl}x_i - J\ddot{\phi}_j x_i dv = \frac{1}{V}\left[\oiint_S (\mu_{kj}x_i)\, n_k ds + \in_{jkl}\int_V \sigma_{kl}x_i dv - \int_V J\ddot{\phi}_j x_i dv\right],\\ &= \frac{1}{V}\left(d_I\int_S \mu_{ij} ds_I + \in_{jkl}\int_V \sigma_{kl}x_i dv - \int_V J\ddot{\phi}_j x_i dv\right).\end{aligned} \tag{11}$$

in which the capital index indicates the same number as its lower case but without summation, and s_I denote the plane with unit normal vector $n_i = 1$. The surface integration of the above equations can be reduced from over enclosed surfaces to a single plane due to each face of the proposed cuboid is normal to the coordinate axis. We introduce $m_j = m_{kj}n_k$ is the couple stress vector acting on a surface for a micropolar medium. Integrating the vector m_j over the surface normal to n_k is the total moment on that surface,

$$M_j^{(s)} = \int_s m_j ds. \tag{12}$$

In classical elasticity for Cauchy medium, the moments on a plane are caused by the non-uniformity of stress distributions, which can be expressed as

$$\begin{cases} M^{\parallel} = M_j^{(i)} = \int_S e_{ijk}\sigma_{II}x_k ds_I, \\ M^{\perp} = M_i^{(i)} = \int_S e_{ijk}\sigma_{ij}x_k ds_I, \end{cases} \tag{13}$$

in which $M^{\parallel}$ and $M^{\perp}$ are the parallel and perpendicular components of moment on the plane, respectively. We may assume that the moments of homogenized model and the unit cell of metamaterial are equivalent on the corresponding planes, hence the first term in the parentheses of Eq. (11) can be further computed and Eq. (11) turns to

$$\bar{\mu}_{ij} = \begin{cases} \frac{1}{V}\left(d_I\int_S e_{ijk}\sigma_{II}x_k ds_I + \in_{jkl}\int_V \sigma_{kl}x_i dv - \int_V J\ddot{\phi}_j x_i dv\right), & \text{for } i \neq j, \\ \frac{1}{V}\left(d_I\int_S e_{imk}\sigma_{ij}x_k ds_I + \in_{jkl}\int_V \sigma_{kl}x_i dv - \int_V J\ddot{\phi}_j x_i dv\right), & \text{for } i = j. \end{cases} \tag{14}$$

Equations (8)–(10) and (14) establish the approximation of average strain, curvature, stress and couple stress from local displacements and stresses which can be conveniently obtained via numerical simulation such as FEM or others. According to these results, the effective parameters $\mathbf{C}^*$ and $\mathbf{D}^*$ could be obtained by solving the following equations

$$\bar{\sigma}_{ij} = C^*_{ijkl}\bar{\varepsilon}_{kl}, \quad \bar{\mu}_{ij} = D^*_{ijkl}\bar{\kappa}_{kl}, \tag{15}$$

while the effective mass density and rotational inertia are given by

$$\rho^{\mathit{eff}} = \frac{1}{V}\frac{F_k}{\bar{\ddot{u}}_k} = \frac{1}{V}\frac{F_k}{-\omega^2\bar{u}_k}, \quad J^{\mathit{eff}} = \frac{1}{V}\frac{M_k}{\bar{\ddot{\phi}}_k} = \frac{1}{V}\frac{M_k}{-\omega^2\bar{\phi}_k}, \tag{16}$$

where

$$F_k = \sum_s F_k^{(s)} = \frac{1}{V}\int_S \sigma_{jk} ds_j, \quad M_k = \sum_s M_k^{(s)} = \frac{1}{V}\int_S \mu_{jk} ds_j, \tag{17}$$

are the resultant force and moment, respectively. Note that if the independent components of **C** or **D** greater than 9, Eq. (15) will not sufficient to solve all material parameters unless extra conditions or constrains are taken into account.

4. PLANE WAVES IN MICROPOLAR MEDIA

Next we discuss the problems for wave propagation in micropolar media. Consider plane wave solutions

$$\left\{ \begin{array}{c} u_i \\ \phi_i \end{array} \right\} = \left\{ \begin{array}{c} U_i \\ \Phi_i \end{array} \right\} \exp\left[i\left(\mathbf{q}\cdot\mathbf{x}-\omega t\right)\right] = \left\{ \begin{array}{c} U_i \\ \Phi_i \end{array} \right\} \exp\left[i\left(q_1x_1+q_2x_2+q_3x_3-\omega t\right)\right], \tag{18}$$

where $U_i(\Phi_i)$ are the displacement(micro-rotation) amplitudes of the harmonic fields, whereas ω and $\mathbf{q}$ are the incident frequency and wave vector, respectively. Substituting Eq. (18) and Eq. (3) into Eq. (2), we obtain

$$\left\{ \begin{array}{l} C_{jikl}q_jq_kU_l - C_{jikl}\in_{klm} q_j\Phi_m = \rho\omega^2U_i, \\ D_{jikl}q_jq_k\Phi_l + \in_{ijk} C_{jkmn}\left(q_mU_n - \in_{mnp}\Phi_p\right) = J\omega^2\Phi_i. \end{array} \right. \tag{19}$$

The above equations can be expanded in matrix form as

$$\left[\begin{array}{cc} \mathbf{\Lambda}_{3\times3} & \boldsymbol{\eta}_{3\times3} \\ \boldsymbol{\xi}_{3\times3} & \mathbf{\Xi}_{3\times3} \end{array} \right] \left\{ \begin{array}{c} \mathbf{U}_{3\times1} \\ \mathbf{\Phi}_{3\times1} \end{array} \right\} = \left\{ \begin{array}{c} \rho\omega^2\mathbf{U} \\ J\omega^2\mathbf{\Phi} \end{array} \right\}, \tag{20}$$

where

$$\begin{array}{l} \Lambda_{il} = q_jq_kC_{jikl}, \quad \eta_{im} = q_j\in_{klm} C_{jikl}, \\ \Xi_{il} = q_jq_kD_{jikl} - \in_{ijk}\in_{mnl} C_{jkmn}, \quad \xi_{in} = q_m\in_{ijk} C_{jkmn}. \end{array} \tag{21}$$

A necessary and sufficient condition for the existence of a non-trivial solution is that the determinant of coefficient matrix vanishes, and it yields

$$\left| \begin{array}{cccccc} \Lambda_{11}-\rho\omega^2 & \Lambda_{12} & \Lambda_{13} & \eta_{11} & \eta_{12} & \eta_{13} \\ \Lambda_{21} & \Lambda_{22}-\rho\omega^2 & \Lambda_{23} & \eta_{21} & \eta_{22} & \eta_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33}-\rho\omega^2 & \eta_{31} & \eta_{32} & \eta_{33} \\ \xi_{11} & \xi_{12} & \xi_{13} & \Xi_{11}-J\omega^2 & \Xi_{12} & \Xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} & \Xi_{21} & \Xi_{22}-J\omega^2 & \Xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} & \Xi_{31} & \Xi_{32} & \Xi_{33}-J\omega^2 \end{array} \right| = 0. \tag{22}$$

The above equation is the dispersion relation of a general anisotropic micropolar medium. Generally, the eigenvalue problem must be solved numerically and all wave propagation modes are coupled. However, for certain high symmetric micropolar media, e.g., isotropic, the wave fields will be decoupled into pure modes so that the closed-form solutions could be obtained.

5. CONCLUSION

In conclusion, we have illustrated a numerical homogenization procedure of a 3D cuboid metamaterial structure. Explicit formulations of averaged physical fields that relates the microscopic local fields to macroscopic quantities are presented in Eqs. (7)–(10) and (14). The general form of dispersion relations of anisotropic metamaterials is demonstrated as well. For further study, it is hope that the presented methodology could be extended to homogenize any periodic metamaterials with unit cells having chiral effects.

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