Solving Nonlinear Helmholtz Equation via Fourier Series

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Abstract— In recent work, we represent a new way of solving non-linear Helmholtz equation. Undefined solution Helmholtz equation is reduced to the system of non-linear algebraic equations by spreading it in a Fourier series, in which we find Fourier coefficients. Later on, by these coefficients we can define our unknown function.

1. INTRODUCTION

Non-linear phenomena accompany many physical processes investigated in optics, acoustics, fluid dynamics, plasma physics and other fields of physics. Abstracting from specific, defined physical properties of mechanisms of processes in each of these areas, it is possible to establish general laws and regularities non-linearity manifestations, regardless of their physical content. This suggests that the theory of non-linear wave processes of physical self-discipline is quite extensive and dynamic. Confirming its widest possibilities is the fact that the methods and results of the theory of non-linear waves have been successfully applied in other fields of knowledge, not only in the natural sciences, but also economic and humanitarian. Significant place in the theory of non-linear waves occupy weakly non-linear wave processes, and increasing interest in them is stimulated by practical needs of optoelectronics, fiber optics and non-linear acoustics [3, 4]. From a physical standpoint weakly non-linear processes distinguished by the condition that the amplitude of the wave field are large enough so that you can not neglect the effects of non-linearity, but they can be seen as complementary to the linear wave background process. Influence of strong non-linearity results in a change of a qualitative nature of the wave process in comparison with the linear case. Analytical description of weak non-linearity requires the development of special methods of asymptotic solutions of the model equations.

In recent work we consider the non-linear Helmholtz equation, modelling weakly non-linear evolution of the beam.

2. THE STATEMENT OF PROBLEM

Consider non-linear Helmholtz equation of next view:

$$\left(k_0^2+\Delta\right)U=-\alpha|U|^2U, \tag{1}$$

where $\alpha \ll 1$, k_0 are constants.

Define Fourier series in complex form

$$U \sim \sum_{n=-\infty}^{+\infty} C_n e^{ixn\frac{2\pi}{l}},$$

where C_n is Fourier coefficients.

Replacing function U by it's Fourier series, we get Helmholtz equation in next form:

$$k_0^2\sum_{n=-\infty}^{+\infty} C_n e^{ixn\frac{2\pi}{l}}-\left(\frac{2\pi}{l}\right)^2\sum_{n=-\infty}^{+\infty} C_n e^{ixn\frac{2\pi}{l}}n^2=-\alpha\left|\sum_{k=-\infty}^{+\infty} C_k e^{ixk\frac{2\pi}{l}}\right|^2\sum_{n=-\infty}^{+\infty} C_n e^{ixn\frac{2\pi}{l}} \tag{2}$$

For further convenience we replace Fourier series by it's partial sums $\sum_{n=-M}^{M} C_n e^{ixn\frac{2\pi}{l}}$, where M is integer and further denote $\frac{2\pi}{l}$ as p.

$$k_0^2\sum_{n=-M}^{M} C_n e^{ixnp}-p^2\sum_{n=-M}^{M} C_n e^{ixnp}n^2=-\alpha\left|\sum_{k=-M}^{M} C_k e^{ixkp}\right|^2\sum_{n=-M}^{M} C_n e^{ixnp} \tag{3}$$

By using mathematical induction to (3), for $M = 1, 2, 3, \ldots$, we obtain formula of Equation (1) for general case:

$$k_0^2 \sum_{n=-M}^{M} C_n e^{ixnp} - p^2 \sum_{n=-M}^{M} C_n e^{ixnp} n^2 = -\alpha \sum_{n=-M}^{M} e^{ipnx} \sum_{s,k=-M}^{M} \bar{C}_{s+k-n} C_s C_k, \tag{4}$$

where $M \to \infty$, $\bar{C}_n$ is conjugate value to C_n.

By using orthonormality

$$\int_0^l e^{impx} e^{-inpx} dx = \delta_{mn}$$

and taking scalar product from both sides of (4), we obtain infinite system of non-linear algebraic equations, which we reduce by M:

$$C_n \left(k_0^2 - p^2 n^2\right) = -\alpha \sum_{s=-M}^{M} \sum_{k=-M}^{M} \bar{C}_{s+k-n} C_s C_k \tag{5}$$

Also get a system of non-linear algebraic equations for conjugate coefficients:

$$\bar{C}_n \left(k_0^2 - p^2 n^2\right) = -\alpha \sum_{s,k=-M}^{M} C_{s+k-n} \bar{C}_s \bar{C}_k \tag{6}$$

Thus the problem is reduced to an infinite system of algebraic Equations (5), (6) for $M \to \infty$.

3. SOLVING A SYSTEM OF NON-LINEAR ALGEBRAIC EQUATIONS

In this part we describe analytical way of solving non-linear system by structured algorithm.

Consider our obtained double size system of non-linear equations with conjugate part. And for convenience we reduce it so, that the number of undefined values coincide with the number of equations.

$$\begin{cases} C_0 k_0^2 = -\alpha \left(\bar{C}_0 C_0^2 + \bar{C}_1 C_0 C_1 + \bar{C}_{-1} C_0 C_{-1} + \ldots\right) \\ C_1 \left(k_0^2 - p^2\right) = -\alpha \left(\bar{C}_{-1} C_0^2 + \bar{C}_0 C_0 C_1 + \bar{C}_{-2} C_0 C_{-1} + \ldots\right) \\ \ldots \\ C_n \left(k_0^2 - p^2 n^2\right) = -\alpha \sum_{s=-M}^{M} \sum_{k=-M}^{M} \bar{C}_{s+k-n} C_s C_k \\ \ldots \\ \bar{C}_1 \left(k_0^2 - p^2\right) = -\alpha \left(C_{-1} C_0^2 + C_0 \bar{C}_0 \bar{C}_1 + C_{-2} \bar{C}_0 \bar{C}_{-1} + \ldots\right) \\ \ldots \\ \bar{C}_n \left(k_0^2 - p^2 n^2\right) = -\alpha \sum_{s,k=-M}^{M} C_{s+k-n} \bar{C}_s \bar{C}_k \end{cases} \tag{7}$$

where $n = \pm 1, \pm 2, \ldots, \pm M$, M is any integer.

Consider iteration method for system. First, we lead (7) to the next form

$$\begin{cases} f_1\left(C_0, \ldots, C_n, \bar{C}_0, \ldots, \bar{C}_n\right) = 0 \\ f_2\left(C_0, \ldots, C_n, \bar{C}_0, \ldots, \bar{C}_n\right) = 0 \\ \ldots \\ f_N\left(C_0, \ldots, C_n, \bar{C}_0, \ldots, \bar{C}_n\right) = 0, \end{cases} \tag{8}$$

where $f_j(C_0, \ldots, C_n, \bar{C}_0, \ldots, \bar{C}_k) : \mathbb{R}^n \to \mathbb{C}$ are non-linear functions, $j = \overline{1, N}$.

1. Set the initial approximation $C^{(0)} = (C_{10}, C_{20}, \ldots, C_{n0})$ and very small value $\varepsilon > 0$, let $k = 0$.
2. Calculate $C^{(k+1)}$ by the formula
$$C^{(k+1)} = \Phi(C^{(k)}),$$
where $\Phi(C) = C + \Lambda F(C)$, $F(C) = (f_1(C), \ldots f_N(C))$, $\Lambda = -W^{-1}(C^{(0)})$ if $\det W(C^{(0)}) \neq 0$,
$$W(C) = \begin{pmatrix} \frac{\partial f_1(C)}{\partial C_0} & \ldots & \frac{\partial f_1(C)}{\partial C_n} & \ldots & \frac{\partial f_1(\bar{C})}{\partial \bar{C}_0} & \ldots & \frac{\partial f_1(\bar{C})}{\partial \bar{C}_n} \\ & \vdots & & \ddots & & \vdots & \\ \frac{\partial f_N(C)}{\partial C_0} & \ldots & \frac{\partial f_N(C)}{\partial C_n} & \ldots & \frac{\partial f_N(\bar{C})}{\partial \bar{C}_0} & \ldots & \frac{\partial f_N(\bar{C})}{\partial \bar{C}_n} \end{pmatrix} - \text{ Jacoby matrix of } N \times N$$

where we take $N = 2n + 2$, $C = (C_0, \ldots, C_n, \bar{C}_0, \ldots, \bar{C}_n)$. In case if $\det W(C^{(0)}) = 0$, then another initial value should be chosen.

3. If $\Delta^{(k+1)} = \max |C_j^{(k+1)} - C_j^{(k)}| \leq \varepsilon$, then process is ended and solution $C^* \cong C^{(k+1)}$. If $\Delta^{(k+1)} > \varepsilon$, then assuming $k = k + 1$ and go to 2.

Denote the right hand side of system (7) respectively $\varphi_1, \varphi_2, \ldots, \varphi_N$.

Theorem 1. (Sufficient condition of convergence of iteration method)
Let functions $\varphi_j(C)$ and $\varphi_j'(C)$ are continuous in G domain, $j = \overline{1, N}$ and the next inequality holds

$$\max_{C \in G} \max_j \sum_{k=0}^{n} \left| \frac{\partial \varphi_j}{\partial C_k} \right| \leq q < 1,$$

where q is constant. If the successive approximations $C^{(k+1)} = \Phi(C^{(k)})$, $k = 0, 1, 2, \ldots,$ do not go out of G, then the process of successive approximations converges: $C^* = \lim\limits_{k \to \infty} C^{(k)}$ and C^* is the unique solution of the system (8) in G.

4. SOME PROPERTY OF SOLUTION OF HELMHOLTZ EQUATION

In this section we led some property of solution of non-linear Helmholtz equation.

Since at least 60 years it is well-known that Nikol'skij-Besov spaces represent a useful notion of regularity not only in the context of approximation theory but also in other branches of mathematics. In fact, for us the scale of Lizorkin-Triebel-Morrey and Nikol'skij-Besov-Morrey spaces will be more important.

Theorem 2. *Let $1 < p,\ q < \infty$, $s > 0$ and $0 \leq \lambda < 1/p$. A function $U \in M_p^\lambda(\mathbb{T})$ belongs to $\mathcal{E}^s_{\lambda,p,q}(\mathbb{T})$ if, and only if,*

$$\| \, U \, |\mathcal{E}^s_{\lambda,p,q} \|^{\#} := \Big\| \, S_1 U \, | M_p^\lambda(\mathbb{T}) \Big\| + \left\| \left(\sum_{N=1}^{\infty} N^{(s-1/q)q} \, | \, U - S_N U \, |^q \right)^{1/q} \Big| M_p^\lambda(\mathbb{T}) \right\| < \infty \,. \tag{9}$$

Furthermore, the quantities $\| \cdot |\mathcal{E}^s_{\lambda,p,q}\|^{\#}$ and $\| \cdot |\mathcal{E}^s_{\lambda,p,q}\|$ are equivalent on $M_p^\lambda(\mathbb{T})$, i.e., there exist two positive constants A, B s.t.

$$A \, \| \, U \, |\mathcal{E}^s_{\lambda,p,q}\|^{\#} \leq \| \, U \, |\mathcal{E}^s_{\lambda,p,q}\| \leq B \, \| \, f \, |\mathcal{E}^s_{\lambda,p,q}\|^{\#}$$

holds for all $U \in M_p^\lambda(\mathbb{T})$.

Remark 1. Of course, Thm. 2 solves problems (i) and (ii) stated at the beginning of the Introduction in [5] with $E = M_p^\lambda(\mathbb{T})$ (under the given restrictions).

Theorem 3. *Let $1 < p < \infty$, $1 \leq q \leq \infty$, $s > 0$ and $0 \leq \lambda < 1/p$. A function $U \in M_p^\lambda(\mathbb{T})$ belongs to $\mathcal{N}^s_{\lambda,p,q}(\mathbb{T})$ if, and only if,*

$$\| \, U \, |\mathcal{N}^s_{\lambda,p,q}\|^{\#} := \Big\| \, S_1 U \, | M_p^\lambda(\mathbb{T}) \Big\| + \left(\sum_{N=1}^{\infty} N^{(s-1/q)q} \, \Big\| \, U - S_N U \, | M_p^\lambda(\mathbb{T}) \Big\|^q \right)^{1/q} < \infty \,. \tag{10}$$

Furthermore, the quantities $\| \cdot |\mathcal{N}^s_{\lambda,p,q}\|^{\#}$ and $\| \cdot |\mathcal{N}^s_{\lambda,p,q}\|$ are equivalent on $M_p^\lambda(\mathbb{T})$, i.e., there exist two positive constants A, B s.t.

$$A \, \| \, U \, |\mathcal{N}^s_{\lambda,p,q}\|^{\#} \leq \| \, U \, |\mathcal{N}^s_{\lambda,p,q}\| \leq B \, \Big\| \, U \, |\mathcal{N}^s_{\lambda,p,q}\Big\|^{\#}$$

holds for all $U \in M_p^\lambda(\mathbb{T})$.

Remark 2. (i) We shall call the spaces $\mathcal{E}^s_{\lambda,p,q}(\mathbb{T})$ periodic Lizorkin-Triebel-Morrey spaces. They represent the Lizorkin-Triebel scale built on the Morrey space $M^\lambda_p(\mathbb{T})$ [5].
(iii) The spaces $\mathcal{N}^s_{\lambda,p,q}(\mathbb{T})$ will be called periodic Nikol'skij-Besov-Morrey spaces. They represent the Nikol'skij-Besov scale built on the Morrey space $M^\lambda_p(\mathbb{T})$ [5].
(iv) We would like to mention that there exist some further scales of smoothness spaces related to Morrey spaces, namely Nikol'skij-Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ as well as Lizorkin-Triebel-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$. (More you may see in [5]).

5. CONCLUSION

We considered non-linear Helmholtz equation, solution is defined via Fourier series and analytical way of solving is represented. Also shortly some properties of solution are led in field of Function spaces and approximation theory.

REFERENCES

1. Vladimirov, V. S., *Equations of Mathematical Physics*, Nauka, Moscow, 1981 (in Russian).
2. Schwartz, L., *Mathematics for the Physical Sciences*, Mir, Moscow, 1965 (in Russian).
3. Ferreira, A. C., M. B. C. Costa, A. G. Colho, Jr., C. S. Sobrinho, J. L. S. Lima, J. W. M. Menezes, M. L. Lyra, A. S. B. Sombra, C. Xu, J. Weng, H. Li, and W. Xiong, "Analysis of the nonlinear optical switching in a Sagnac interferometer with non-instantaneous Kerr effect," *Optics Communications*, No. 5, 23–38; Vol. 285, No. 6, 1408–1417, 2012.
4. Mukherjee, P. K., "Pretransitional Kerr effect and nonlinear dielectric effect in the isotropic phase of the isotropic to smectic-E phase transition," *Journal of Molecular Liquids*, Vol. 175, 1–3, 2012.
5. Sautbekova, M. and W. Sickel, "Strong summability of Fourier series and Morrey spaces," *Analysis Mathematica*, Vol. 40, 31–62, 2014.