Analysis of Probability Distribution of Inverse Problem of Nonlinear Model

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Abstract— In the remote sensing field, the SNR is very low and the model is nonlinearity usually. Thus, it is important to analyze the impact of model nonlinearity. Probability distribution can be used to analyze the parameter distribution directly. Some issues involving priors and posteriors were proposed for models with significant nonlinearity and low signal to noise ratio (SNR). Two important issues are as follows: (1) The unimodal probability density function (PDF) of the observation quantity can give rise to a multi-modal PDF of the parameter (parameter). This property could lead to an incorrect maximum a posteriori estimate or a biased mean central estimate. It means that the biased evaluation results could be derived from statistical methods; (2) The rules for assigning non-informative priors in the measurement approach are different from the principle of maximum entropy. The authors point out the distinctions between PDF and probability distribution of non-linear models in the parameter space based on the resolution relationship between the observation space and the parameter space. For models with significant nonlinearity, if the unit grid interval of the observation space is considered regular, then the unit grid interval of the parameter space is considered irregular. The distribution obtained from regular grid is an approximation to the PDF. The probability should be calculated by integrating the PDF in the corresponding irregular grid. The difference between PDF and probability distribution gives rise to the difficulty in reverse problems. Analyzing the properties of the parameter by using the probability distribution instead of the PDF in the parameter space is necessary. The corresponding relationship between the observation and the parameter is researched based on resolution limit analysis. The non-uniform prior PDF was derived from uniform prior probability distribution in accordance with the principle of maximum entropy. Nonlinear analysis of the nature of the probability density distribution is necessary to eliminate bias caused by statistical methods.

1. INTRODUCTION

The methods for solving reverse problems is important in oceanic and atmospheric science. Many papers [1, 2] have applied the observation equation approach and the measurement equation approach to produce the PDF.

We assume that a non-informative prior is obtained, that the model $\eta=\phi(\alpha,\ \beta)$ represents a smooth response with respect to α, and that $\phi'(\alpha,\ \beta)$ is continuous. Given an indication ζ, he measurement equation approach determines the PDF $p_{ME}(\alpha,\ \beta)$ as

$$p_{ME}(\alpha, \beta|\zeta) \propto p(\zeta|\alpha, \beta)\phi'(\alpha, \beta)p(\beta) \tag{1}$$

which involves the likelihood $p(\zeta/\alpha,\ \beta)$ and the particular prior

$$p(\alpha, \beta) \propto \phi'(\alpha, \beta)p(\beta). \tag{2}$$

Although these recent studies have addressed the effect of the parameter through a nonlinear model, they have not offered an explicit analysis of the relationship between model nonlinearity and the properties of PDF and probability distribution. Several issues have been raised for models with significant nonlinearity [3–6], the two important issues are: 1) The assignment of the non-informative prior (2) is different from the set of rules stated in GUMS1. 2) Explaining some properties of the posterior PDF is difficult. For instance, a unimodal PDF of the observation quantity can give rise to a multi-modal PDF of the parameter in a nonlinear monotonic model. This property could lead to the incorrect maximum a posteriori (MAP) estimate or maximum likelihood estimate in practice if the global maximum is desired.

We take the cubic model $\eta = \alpha^3$ as an example.

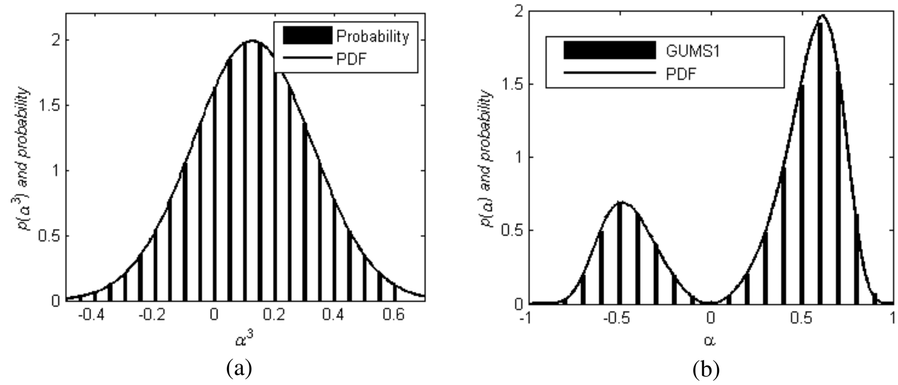

Figure 1: Comparison of PDFs and probability distributions in the observation space and in the parameter space from the model $\eta = \alpha^3$ for the case $\zeta = 0.5^3$ and $\sigma = 0.2$. The bars correspond to the probability distributions from GUMS1, the line corresponds to the result of PDF. (a) The PDF and probability distribution of the observation quantity is normal distribution. (b) The posterior PDF and probability distribution of the parameter is double-modal distribution.

Figure 1 shows the comparison results of the PDFs. The cubic function is a one-to-one correspondence function. The normal PDF of the observation quantity gives rise to a double-modal PDF of the parameter. Thus, this property could lead to an incorrect MAP estimate. A paradox is created with the requirement of parametrization invariance, which states that intrinsic properties of an object do not depend on any particular choice of parameters. In addition, it is difficult to estimate the parameter in non-linear function from the posterior PDF. These issues are fundamental in measurement uncertainty evaluation.

2. ANALYSIS OF THE PROBABILITY DISTRIBUTION

When the instrument resolution is known, defining a discrete and regular grid in the observation space as $\eta = (\eta_{\min}, \ldots, \eta_i, \ldots, \eta_{\max})$ is possible. The observation quantity resolution is denoted by Δd, given that $\Delta d = \text{const}$, so that a displayed reading η_i implies that the actual signal lies the interval $[\eta_i - (\Delta d/2), \eta_i + (\Delta d/2)]$. Within the interval, the relationship of the probability distribution and the PDF in the observation space is as follows:

$$P_{OE}(\eta_i) = \int_{\eta_i - \Delta d/2}^{\eta_i + \Delta d/2} p_{OE}(\eta_i) d\eta. \tag{3}$$

Here, $P_{OE}(\eta_i)$ denotes the probability of i-th observation quantity, and $p_{OE}(\eta_i)$ denotes the PDF in the i-th interval in the observation space.

The parameter is determined by measurement equation $\alpha = \psi(\eta, \beta)$. By this transformation the nonlinear functional dependence of the observation quantities on parameters is mapped. This is equivalent to producing a graph of a function on a coordinate grid. The grid in the measurement space $\alpha = (\alpha_{\min}, \ldots, \alpha_i \ldots, \alpha_{\max})$ is defined as $\Delta\alpha_i$. A indication α_i implies that the actual signal lies the interval $[\psi(\eta_i - \Delta d/2), \psi(\eta_i + \Delta d/2)]$. Owing to the $\Delta d = \text{const}$ and nonlinearity, the unit grid is irregular. The relationship of the probability distribution and the PDF in the parameter space is as follows:

$$P_{ME}(\alpha_i) = \int_{\psi(\eta_i - \Delta d/2}^{\psi(\eta_i + \Delta d/2)} p_{ME}(\alpha_i) d\alpha \tag{4}$$

where $P_{ME}(\alpha_i)$ denotes the probability of the i-th parameter, and $p_{ME}(\alpha_i)$ denotes the PDF in the ith interval in the parameter space. From (4), we can observe that the distinction appears between the PDF and the probability distribution in the irregular grid parameter space. The properties of the parameter should be analyzed by using the probability distribution instead of the PDF in the parameter space. For a one-to-one function, the mapping relationship of the probability distribution between the two spaces is obtained by combining (3) and (4).

$$P_{ME}(\eta_i) = P_{OE}(\alpha_i) \tag{5}$$

Relationship (5) states that the probability of the observation quantity is equal to the probability of the parameter.

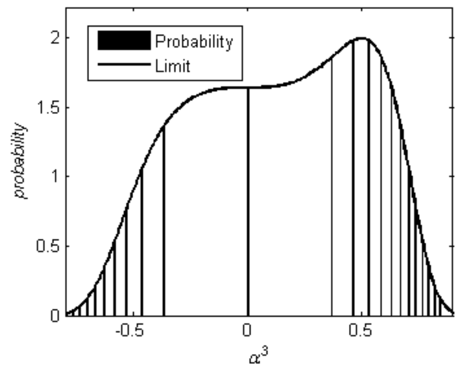

Figure 2: Posterior probability distribution of the parameter obtained by the modified Monte Carlo method for the model $\eta = \alpha^3$ with $\Delta d = 0.05$, $\zeta = 0.5^3$ and $\sigma = 0.2$ corresponding the observation quantity has normal distribution.

Figure 2 shows the probability distribution result of the parameters for the cubic function model. Corresponding to the unimodal probability distribution of the observation quantity, the probability distribution of the parameter is also unimodal. From the Figure 1(a) and Figure 2, we can conclude that the mapping relationship of the probability distribution is invariant between the two spaces. The MAP probability distribution estimate is invariant in both the observation space and in the parameter space.

3. CONCLUSIONS

We pointed out the difference between PDF and probability distribution in the parameter space based on the resolution relationship between the observation space and the parameter space.

Analyzing the properties of the parameter by using the probability distribution instead of the PDF in the parameter space is necessary. The corresponding relationship between the observation and the parameter was obtained based on resolution limit analysis. The non-uniform prior PDF was derived from uniform prior probability distribution in accordance with the rules stated in GUMS1.

Probability distribution can be used to analyze the parameter distribution directly. The method of the MAP probability distribution is useful one to reverse parameter in Inverse Problem.

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