On the Treatment of Hypersingularity for Solving Volume Integral Equations

P. C. Wang, Z. G. Zhou, J. H. Zhou, X. F. Yin, and M. S. Tong
Department of Electronic Science and Technology, Tongji University
4800 Cao'an Road, Shanghai 201804, China

Abstract— In the meshless scheme for solving volume integral equations (VIEs), one needs to evaluate hypersingular integrals over a small cylindrical domain excluded from the whole domain for the sake of regularizing integrands. Those integrals can be evaluated semi-numerically but the implementation may not be convenient. In this work, we present some new formulas which are purely analytical for those integrals so that their evaluation can be greatly simplified. A numerical example for electromagnetic (EM) scattering is presented to demonstrate the new formulas.

1. INTRODUCTION

Volume integral equations are indispensable for solving electromagnetic (EM) problems with inhomogeneous or anisotropic media by an integral equation approach [1]. The VIEs are usually solved by the method of moments with the divergence-conforming Schaubert-Wilton-Glisson (SWG) basis function [2] or curl-conforming edge basis function [3]. These basis functions require conforming meshes for geometrical discretization, resulting in a high cost for preprocessing. Aiming to reduce the cost of meshing or remeshing geometries, point-matching methods like Nyström method and meshless methods were proposed and have been widely used in practice [4–12]. In these methods, we need to accurately evaluate singular volume integrals involving derivatives of the Green's function. Although many techniques have been developed for evaluating weakly and strongly singular integrals, the techniques for evaluating hypersingular integrals which come from the double gradient of the dyadic Green's function are very limited. Previously, we developed a treatment technique which uses a numerical integration for line integrals based on the analytical formulas of surface integrals [12]. The treatment is valid but it may not be convenient in implementation. In this work, we present new formulas for evaluating the hypersingular volume integrals over a cylindrical domain so that the implementation can be greatly simplified. We demonstrate the new formulas by solving an EM scattering problem and good results have been observed.

2. VOLUME INTEGRAL EQUATIONS

Consider the EM scattering by a dielectric (nonmagnetic) object which has a relative permittivity ϵ_r and is embedded in the free space with a permittivity ϵ_0 and a permeability μ_0. The VIE for describing the problem can be written as [1]

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{inc}(\mathbf{r}) + i\omega\mu_0 \int_V \bar{\mathbf{G}}\left(\mathbf{r},\mathbf{r}'\right) \cdot \mathbf{J}_V\left(\mathbf{r}'\right) d\mathbf{r}', \quad \mathbf{r} \in V \tag{1}$$

where $\mathbf{E}^{inc}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ are the incident electric field and the total electric field inside the object, respectively, and $\mathbf{J}_V(\mathbf{r}') = i\omega\epsilon_0(1-\epsilon_r)\mathbf{E}(\mathbf{r}')$ is the volumetric current density inside the object. Also, the integral kernel is the dyadic Green's function defined by

$$\bar{\mathbf{G}}\left(\mathbf{r},\mathbf{r}'\right) = \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k_0^2}\right) g\left(\mathbf{r},\mathbf{r}'\right) \tag{2}$$

where $\bar{\mathbf{I}}$ is the identity dyad, $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the free-space wavenumber, and $g(\mathbf{r},\mathbf{r}') = e^{ik_0R}/(4\pi R)$ is the scalar Green's function with $R = |\mathbf{r}-\mathbf{r}'|$ being the distance between an observation point $\mathbf{r}$ and a source point $\mathbf{r}'$. The above VIE can be solved by a meshless method as described in [12]. The method transforms the volume integral into a boundary or surface integral based on the Green-Gauss theorem so that discretizing the volume domain can be avoided, leading to a truly meshless scheme. However, this transformation is only valid for a regular integrand while the integral kernel of the VIE is hypersingular. We need to regularize the integral kernel by excluding a small cylindrical domain V_0 enclosing an observation node and the volume integral over the small cylindrical domain must be specially treated.

3. EVALUATION OF HYPERSINGULAR VOLUME INTEGRALS

The hypersingular volume integrals result from the double gradient of the dyadic Green's function and take the following form after performing a singularity subtraction

$$I_1 = \int_{V_0} \left(\frac{3u^2}{R^5} - \frac{1}{R^3} \right) dV, \quad I_2 = \int_{V_0} \left(\frac{3v^2}{R^5} - \frac{1}{R^3} \right) dV \tag{3}$$

$$I_3 = \int_{V_0} \left(\frac{3w^2}{R^5} - \frac{1}{R^3} \right) dV, \quad I_4 = \int_{V_0} \frac{uv}{R^5} dV \tag{4}$$

$$I_5 = \int_{V_0} \frac{w(-u)}{R^5} dV, \quad I_6 = \int_{V_0} \frac{w(-v)}{R^5} dV \tag{5}$$

where we have established a local coordinate system (u, v, w) as shown in Figure 1 for the integrals and the observation node is located at the origin or the center of the cylindrical domain with a radius a and a height $2h$ while the source point is located at (u, v, w). The evaluation of those hypersingular integrals requires a numerical integration for line integrals in our original work [12] and this may not be very convenient in applications. Now we present new formulas which are purely closed-form for those integrals so as to facilitate the implementation of the meshless method. In the cylindrical domain, we also establish a polar coordinate system (ρ, ϕ) over its cross section, thus we have $\rho^2 = u^2 + v^2$ and $R^2 = \rho^2 + w^2$. The differential of cylindrical volume domain is $\rho d\rho d\phi dw$ but the integration over ϕ is just 2π because there is no dependence upon ϕ in the integrands. The above hypersingular integrals can be evaluated by the following integrals for which we have derived closed-form expressions.

$$\int_{V_0} \frac{dV}{4\pi R^3} = \int_0^h \int_0^a \frac{\rho d\rho dw}{(\rho^2 + w^2)^{3/2}} = -\ln \left[\frac{1}{w} \left(w + \sqrt{w^2 + a^2} \right) \right] \Big|_0^h \tag{6}$$

$$\int_{V_0} \frac{u^2 dV}{4\pi R^3} = \int_0^h \int_0^a \frac{\rho^3 d\rho dw}{2(\rho^2 + w^2)^{3/2}} = \frac{h}{2} \left[\sqrt{a^2 + h^2} - h \right] \tag{7}$$

$$\int_{V_0} \frac{w^2 dV}{4\pi R^3} = \int_0^h \int_0^a \frac{\rho w^2 d\rho dw}{2(\rho^2 + w^2)^{3/2}} = \frac{1}{2} \left\{ h^2 - h\sqrt{a^2 + h^2} + a^2 \ln \left[\frac{h}{a} + \sqrt{1 + \frac{h^2}{a^2}} \right] \right\} \tag{8}$$

$$\int_{V_0} \frac{u^2 dV}{4\pi R^4} = \int_0^h \int_0^a \frac{\rho^3 d\rho dw}{2(\rho^2 + w^2)^2} = \frac{1}{4} \left[h \ln \left(1 + \frac{a^2}{h^2} \right) + a \tan^{-1} \left(\frac{h}{a} \right) \right] \tag{9}$$

$$\int_{V_0} \frac{w^2 dV}{4\pi R^4} = \int_0^h \int_0^a \frac{\rho w^2 d\rho dw}{(\rho^2 + w^2)^2} = \frac{a}{2} \tan^{-1} \left(\frac{h}{a} \right) \tag{10}$$

$$\int_{V_0} \frac{u^2 dV}{4\pi R^5} = \int_0^h \int_0^a \frac{\rho^3 d\rho dw}{2(\rho^2 + w^2)^{5/2}} = -\frac{1}{6} \left\{ 2\ln \left[1 + \sqrt{1 + \frac{a^2}{w^2}} \right] + \frac{w}{\sqrt{a^2 + w^2}} \right\} \Big|_0^h \tag{11}$$

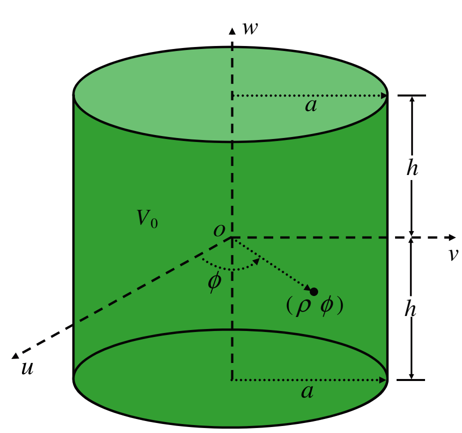

Figure 1: Cylindrical volume domain for evaluating hypersingular integrals.

$$\int_{V_0} \frac{w^2 dV}{4\pi R^5} = \int_0^h \int_0^a \frac{\rho w^2 d\rho dw}{(\rho^2+w^2)^{5/2}} = \frac{1}{3}\left\{\frac{w}{\sqrt{a^2+w^2}} - \ln\left[1+\sqrt{1+\frac{a^2}{w^2}}\right]\right\}\Bigg|_0^h. \tag{12}$$

Some integrals in the above are individually divergent when $w \to 0$, but their combination will cancel the divergent terms, leading to convergent expressions, i.e.,

$$\int_{V_0} \frac{3u^2 - R^2}{4\pi R^5} dV = -\frac{h}{2\sqrt{a^2+h^2}} \tag{13}$$

$$\int_{V_0} \frac{3w^2 - R^2}{4\pi R^5} dV = \frac{h}{\sqrt{a^2+h^2}}. \tag{14}$$

We have omitted some integrals duo to their similarity and the above integrals are representative. We have also incorporated the results of integration over ϕ and reduced the integral interval over w due to its symmetry.

4. NUMERICAL EXAMPLE

We demonstrate the new formulas by solving an EM scattering problem in which the scatterer is a homogeneous dielectric sphere with a radius r and a relative permittivity ϵ_r. It is assumed that the incident wave is a plane wave with a frequency $f = 300\,\text{MHz}$ and is propagating along the $+z$ direction. We calculate the bistatic radar cross section (RCS) observed along the principal cut ($\phi = 0°$ and $\theta = 0° - 180°$) for the scatterer in both vertical polarization (VV) and horizontal polarization (HH). Figure 2 shows the solutions when $r = 0.5\lambda$ and $\epsilon_r = 4.0$ and we can see that the solutions agree with the exact Mie-series solutions very well.

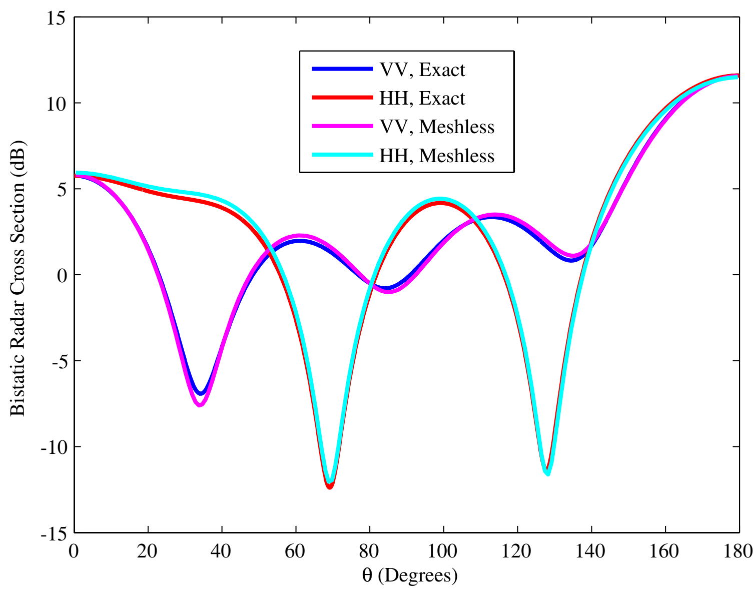

Figure 2: Bistatic RCS solutions for EM scattering by a dielectric sphere with a radius $r = 0.5\lambda$ and relative permittivity $\epsilon_r = 4.0$.

5. CONCLUSION

The accurate evaluation of hypersingular integrals is essential for using meshless methods to solve EM problems. Though they can be evaluated by performing a line integral numerically based on the analytical formulas of surface integrals, the implementation may not be convenient. In this work, we present purely analytical formulas as an alternative for evaluating those integrals and they can be easily used. A numerical example for solving EM scattering problems has been presented to verify the formulas.

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