Radiation of Inverted Pendulum with Hysteretic Nonlinearity

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Abstract— In this work we investigate the radiation of the charged inverted pendulum with a hysteretic nonlinearity in the form of a backlash in the suspension point. The radiated power is obtained in the frame of dipole approach. Due to the presence of hysteretic nonlinearity there are some interesting peculiarities take place, in particular, our numerical simulation shows that the radiated power turns in to zero during the hysteretic nonlinearity acts. This fact opens a new way for control of the inverted pendulum's dynamics.

1. INTRODUCTION

The problem of the inverted pendulum has a long history [1, 2] and remains relevant even in the present days [3–5]. As well known the model of the inverted pendulum plays the central role in the control theory [6–11]. It is well established benchmark problem that provides many challenging problems to control design. Because of their nonlinear nature pendulums have maintained their usefulness and they are now used to illustrate many of the ideas emerging in the field of nonlinear control [12]. Typical examples are feedback stabilization, variable structure control, passivity based control, back-stepping and forwarding, nonlinear observers, friction compensation, and nonlinear model reduction. The challenges of control made the inverted pendulum systems a classic tools in control laboratories.

In order to make an adequately description of the dynamics of real physical and mechanical systems it is necessary to take into account the effects of hysteretic nature such as "backlashes", "stops", etc. The mathematical models of such nonlinearities according to the classical patterns of Krasnosel'skii and Pokrovskii [13], reduce to operators that are treated as converters in an appropriate function spaces. The dynamics of such converters are described by the relation of "input-state" and "state-output".

As is known, most of the real physical and technical systems contain a various kind of parts that can be represented as a cylinder with a piston. Inevitably, the backlashes appear in such systems during its long operation due to the "aging" of the materials. As was mentioned above such backlashes are of hysteretic nature and the analysis of such nonlinearities is quiet important and actual problem. Also we would like to note that, for our knowledge, the problem of charged inverted pendulum was not considered in the literature. In this way the problem of stabilization of inverted pendulum using, e.g., an electrical field, seems novel and promisingly.

2. INVERTED PENDULUM WITH HYSTERETIC NONLINEARITY IN SUSPENSION: MATHEMATICAL MODEL

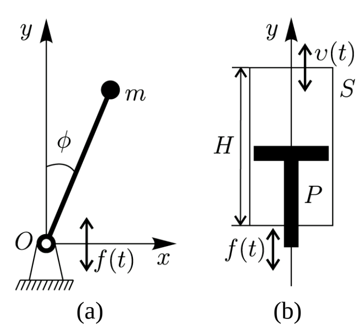

Figure 1: Geometry of the problem. (a) General view of the inverted pendulum. (b) The suspension point (cylinder and piston).

The model of inverted pendulum with oscillating suspension point (see panel a in Fig. 1) was studied in detail by Kapitza [2]. Let us recall that the equation of motion of pendulum has the form:

$$\ddot{\phi} - \frac{1}{l}[g + \ddot{f}(t)]\sin\phi = 0 \tag{1}$$

where ϕ is the angle of vertical deviation of the pendulum, l is the pendulum's length, g is the gravitational acceleration and $f(t)$ is the law of motion of the suspension point (of course, this equation should be considered together with the corresponding initial conditions).

Let us consider a system where the base of the pendulum is a physical system (P, S) formed by a cylinder of length H and the piston P[1]. We determine the piston's position by the coordinate $f(t)$ and the cylinder's position by coordinate $v(t)$. Let us assume also that the "leading" element in the system (P, S) is a cylinder P. In this assumption the system (P, S) can be considered as a converter Γ with the input signal $f(t)$ (piston's position) and the output signal $v(t)$ (cylinder's position). Such a converter is called *backlash*. The set of its possible states is $f(t) \le v(t) \le f(t)+H$ $(-\infty < f(t) < \infty)$. The cylinder's position $v(t)$ at $t > t_0$ is defined by $v(t) = \Gamma[t_0, v(t_0)]f(t)$, where $\Gamma[t_0, v(t_0)]$ is the operator defined for each $v_0 = v(t_0)$ on the set of continuous inputs $f(t)$ $(t > t_0)$ for which $v_0 - H < f(t) < v_0$ [13].

We assume that the piston's acceleration periodically changes from $-a\omega^2$ to $a\omega^2$ with the frequency ω. This assumption consists in the fact that the linearized equation of motion of such a pendulum can be written in the form[2]:

$$\begin{aligned} &\ddot{\phi} - \frac{1}{l}[g + a\omega^2 G(t,H)w(t)]\phi = 0, \\ &w(t) = -\text{sign}[\sin(\omega t)], \\ &\phi(0) = \phi_{10},\ \dot{\phi}(0) = \phi_{20}, \end{aligned} \tag{2}$$

where $\text{sign}(z)$ is the usual signum function, $G(t,H)w(t)$ is the acceleration of the suspension point and

$$G(t,H) = \begin{cases} 0, & t \in (t^*, t^* + \Delta t), \\ 1, & t \text{ out of } (t^*, t^* + \Delta t), \end{cases}$$

where t^* are the moments after which the acceleration's sign change takes place, $\Delta t = \sqrt{\frac{2H}{a\omega^2}}$ is the time for which the piston passes through the cylinder.

Let we pass to dimensionless units in (2) using the following change:

$$x \equiv \phi,\ \tau = \omega t,\ k = \frac{g}{l\omega^2},\ s = \frac{a}{l},\ \Delta\tau = \sqrt{\frac{2H}{sl}}.$$

As a result, we obtain an equation similar to Meissner equation, but with the negative coefficients and hysteretic nonlinearity, namely:

$$\begin{aligned} &\ddot{x} - [k - sG(\tau,H)\text{sign}(\sin\tau)]x = 0, \\ &G(\tau,H) = \begin{cases} 0, & \tau \in (\tau^*, \tau^* + \Delta\tau), \\ 1, & \tau \text{ out of } (\tau^*, \tau^* + \Delta\tau), \end{cases} \\ &x(0) = x_{10},\ \dot{x}(0) = x_{20}, \end{aligned} \tag{3}$$

Using the monodromy matrix technique, based on the Floquet results, we can determine the

[1]Both the cylinder and piston are ideal, absolutely rigid and can move along the y-axis in the infinite ranges as it is shown in panel b of the Fig. 1.

[2]It should also be pointed out that such a periodic behavior of the piston's acceleration (namely, the fact that the acceleration of the piston changes from $-a\omega^2$ to $a\omega^2$) is an assumption of the model presented in this paper. Such a model allows us to obtain some analytical results, namely, the explicit conditions for the stability zones. Also, the numerical simulations are most effectively in the frame of this model. Moreover, such a model of the piston's behavior most effectively and adequately describes the dynamics of the parts of real technical devices.

zones of stabilization for the system under consideration (for details see [5]), namely:

$$\Bigg|\cos(k_2\gamma)\left[2\cosh(2\sqrt{k}\Delta\tau)\cosh(k_1\gamma)+\sinh(2\sqrt{k}\Delta\tau)\sinh(k_1\gamma)\left(\frac{\sqrt{k}}{k_1}+\frac{k_1}{\sqrt{k}}\right)\right]$$
$$+\sin(k_2\gamma)\left[\sinh(2\sqrt{k}\Delta\tau)\cosh(k_1\gamma)\left(\frac{\sqrt{k}}{k_2}-\frac{k_2}{\sqrt{k}}\right)+\cosh^2(\sqrt{k}\Delta\tau)\sinh(k_1\gamma)\left(\frac{k_1}{k_2}-\frac{k_2}{k_1}\right)\right.$$
$$\left.+\sinh^2(\sqrt{k}\Delta\tau)\sinh(k_1\gamma)\left(\frac{k}{k_1k_2}-\frac{k_1k_2}{k}\right)\right]\Bigg|<2. \tag{4}$$

Here $(k_1)^2 = k + s$, $(k_2)^2 = s - k$ $(s > k)$, $\gamma = \pi - \Delta\tau$. Thus, the *stability zone* of the system (3) in the space of parameters is defined by the inequality (4).

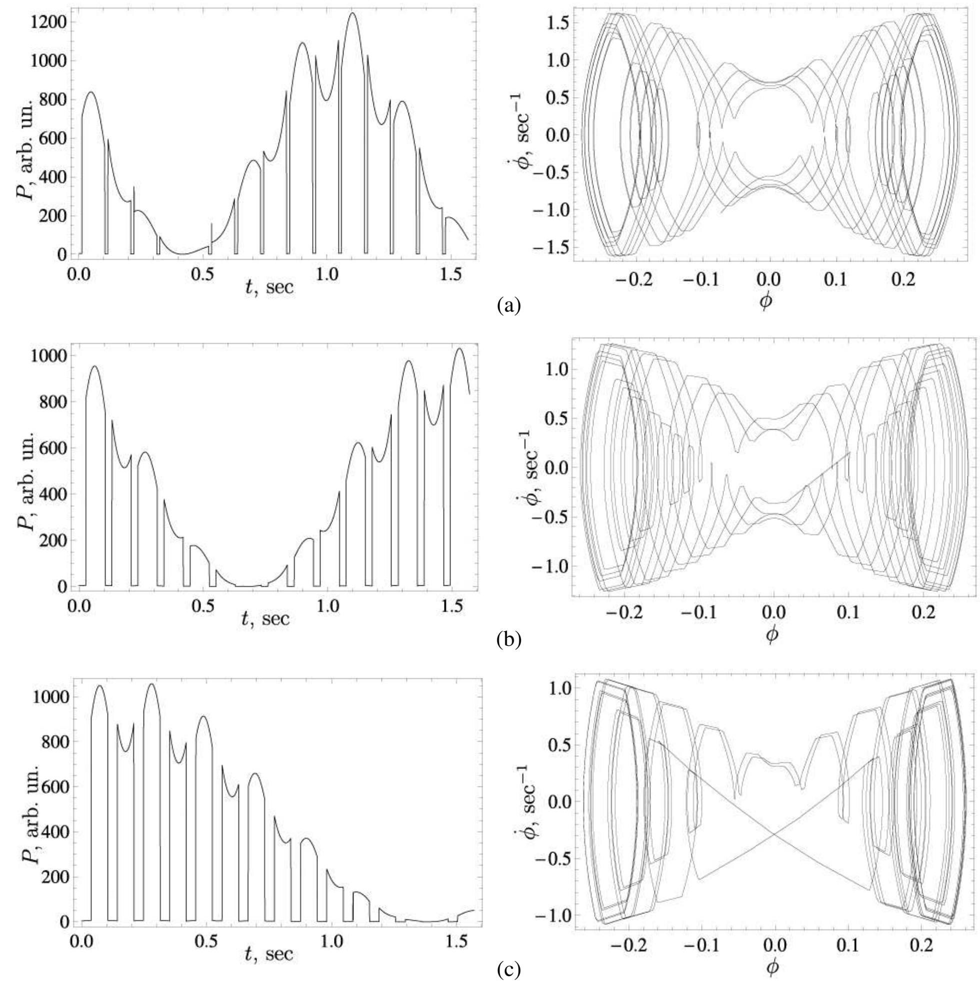

Figure 2: Radiated power P (per arbitrary units) as a function of time and phase portrait of inverted pendulum with the hysteretic nonlinearity in suspension for various values of hysteretic parameter H (another parameters of the system under consideration are presented in the main text). (a): $H = 0.01\,\mathrm{m}$; (b): $H = 0.05\,\mathrm{m}$; (c): $H = 0.1\,\mathrm{m}$.

3. RADIATION OF INVERTED PENDULUM WITH HYSTERETIC NONLINEARITY

Consistent numerical solution of the Equation (3) allows us to determine the law of motion $\phi(t)$. Let us assume also that the pendulum is charged (the charge of a pendulum is q). Following the ideology of classical electrodynamics this fact means that the system under consideration can be considered as a source of electromagnetic radiation (because of non zero acceleration). Namely, the radiated power in the dipole approach can be written in the standard manner [14] (of course, for such a system the dipole momentum is non zero and, as a result, the dipole radiation takes place):

$$P = \frac{2}{3c^3}|\ddot{d}(t)|^2 = \frac{2q^2l^2}{3c^3}|\ddot{\phi}(t)|^2, \tag{5}$$

where $d(t) = ql\phi(t)$ is a dipole momentum of a pendulum (as usual, double dot denotes the second derivative), c is the speed of light.

In the Fig. 2, we present the results of numerical simulations for the radiated power (per arbitrary units, namely we present the value of $\frac{3c^3P}{2q^2l^2}$) togeteher with the corresponding phase portrait at various values of hysteretic parameter H. The parameters of pendulum are: $l = 1\,\mathrm{m}$, $g = 9.8\,\mathrm{m}\cdot\mathrm{sec}^{-2}$; the amplitude and frequency of oscillation of the piston are $a = 0.15\,\mathrm{m}$ and $\omega = 30\,\mathrm{sec}^{-1}$, respectively; the initial conditions are $\phi(0) = 0.2$ and $\dot{\phi}(0) = 1\,\mathrm{sec}^{-1}$.

As we can see the presence of the hysteretic nonlinearity leads to the fact that the radiated power turns in to zero at some time moments. These moments exactly correspond to the time intervals during the hysteretic nonlinearity acts. In this way, this fact opens a new way for control of the inverted pendulum's dynamics, e.g., using the electrical field.

4. CONCLUSIONS

In this paper we have considered the problem of inverted pendulum with a hysteretic nonlinearity in the form of a backlash in suspension point. Namely, we have pointed out on the possibility to detecting of the electromagnetic radiation from the charged pendulum. Due to the presence of hysteretic nonlinearity there are some interesting results take place. In particular, our numerical simulations show that the radiated power turns in to zero at the time moments during the hysteretic nonlinearity acts. This fact opens a new way for control of the dynamics of inverted pendulum (e.g., using the electrical field or, in general, the fields with electromagnetic nature). Of course, the problem of stabilization of the inverted pendulum using an electrical field seems novel and promisingly.

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