Higher-order Surface Modes in the Goubau Line

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Abstract— Existence of the higher-order surface waves of the Goubau line is proved and their structure is analyzed. An efficient computational approach is proposed based on numerical solution to initial-value problems obtained by the parameter differentiation. Several applications and further research directions are discussed.

1. INTRODUCTION

A significant amount of fundamental results of the mathematical theory of wave propagation in open and shielded metal-dielectric waveguides obtained recently [1, 2] open new possibilities for solving forward and inverse problems of electromagnetics. In particular, the description of the spectrum of normal waves is completed for a wide family of waveguides with dielectric inclusions (which actually envelopes all known types of basic guiding structures including slot and strip lines). It is shown that the spectrum is nonempty, forms a countable set of isolated points on the complex plane (there is no continuous spectrum), is located symmetrically in a certain band, and the spectral points enter the spectrum in fours. Consequently, the mathematical background and potential related to comprehensive mathematical modeling and analysis of more complicated structures increase. In this work, we apply some of these results to investigate higher-order surface modes propagating in an open metal-dielectric waveguide, the Goubau line (GL): a perfectly conducting cylinder of circular cross section covered by a concentric dielectric layer (see Fig. 1).

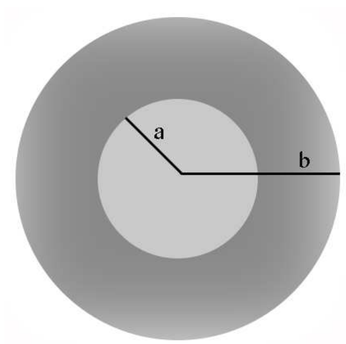

Figure 1: The Goubau line, where a and b are the radii of the internal (perfectly conducting) and external (dielectric) cylinders.

A. Sommerfeld predicted [3] existence of a wire wave propagating along a conducting cylinder. The similar type of wave is known for a GL introduced by Goubau [4]. This wave propagates outside a cylindrical conducting surface, which is possible if the surface has a coating, and is a surface mode; a thin dielectric cover plays the role of conductivity. A GL allows thus to transmit electromagnetic waves along a single wire when non-radiating surface waves are bound to the wire. Since the early 1950s, a big variety of studies and applications of GLs and covered cylindrical structures have been performed and published; one may find a comprehensive list of references, e.g., in [5–7]; many important applications, including the cloaking, are outlined in [7, 8]. However, rigorous proofs of the existence of surface waves and a satisfactory mathematical analysis of the GL multi-parameter dispersion equations (DEs) are absent, to the best of our knowledge. In particular, very little information is available concerning the existence, properties, and structure of higher-order surface waves in a GL.

In this paper, we show that analysis of the spectrum of higher-order surface modes reduces to the solution of specific families of Sturm-Liouville boundary eigenvalue problems with the boundary operator depending on the spectral parameter. We prove the existence and analyze the spectrum of higher-order surface modes revealing some of their important properties that were not reported earlier. An efficient computational approach is proposed using numerical solution to the appropriate initial-value problems. A comparison with dielectric guides (fibers) shows advantages of GLs as transmission lines in certain frequency ranges. The developed techniques enable one to study conducting cylinders, complex waves, and multi-layer lossy dielectric covers.

2. STATEMENT

Consider the propagation of symmetric eigenwaves, described in terms of nontrivial solutions to homogeneous Maxwell's equations in the GL, in which a and b, $b > a$, are the radii of the internal (perfectly conducting) and external (dielectric) cylinders. We consider symmetric azimuthally-independent waves having the nonzero components

$$\mathbf{H} = [0, H_2(r,z), 0], \quad \mathbf{E} = [E_1(r,z), 0, E_3(r,z)], \tag{1}$$

$$E_1 = \phi(r)e^{-i\beta z}, \quad H_2 = -\frac{i\omega\epsilon}{k_s^2}\frac{d\phi}{dr}e^{-i\beta z}, \quad E_3 = -\frac{i\beta}{k_s^2}\frac{d\phi}{dr}e^{-i\beta z}, \tag{2}$$

where

$$k_s^2 = \begin{cases} k_0^2 - \beta^2, & r > b, \\ \epsilon k_0^2 - \beta^2, & a < r < b, \end{cases}$$

β is the wave propagation constant (spectral parameter), $\epsilon > 1$ is relative permittivity of the homogeneous dielectric, k_0 is the free-space wavenumber, and ϕ is a cylindrical function. Determination of surface waves (2) can be reduced to the following singular Sturm-Liouville boundary eigenvalue problems on the half-line with a discontinuous (piecewise constant) coefficient in the differential equation:

$$\mathcal{L}\phi \equiv \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + k_s^2\phi = 0,\ r > a,\ \phi(a) = 0,\ [\phi]|_{r=b} = \left[\frac{d\phi}{dr}\right]\bigg|_{r=b} = 0,\ \phi(r) \to 0,\ r \to \infty, \tag{3}$$

$$\phi \in C^1[a, +\infty) \cap C^2(a,b) \cap C^2(b, +\infty). \tag{4}$$

The conditions at infinity in (3) and the form (2) can be taken as a definition of symmetric (surface) waves in open waveguides under study. The surface waves are described in terms of real-valued quantities; in particular, the boundary operators in (3) is defined (as real-valued functions of a real variable γ or $\lambda = \gamma^2$) on a certain interval I. The spectrum of surface waves may be empty or consist of several (real) points located on this interval. For surface waves, potential function $\phi(r)$ can be represented, according to conditions at infinity, as

$$\phi(r) = \begin{cases} AK_0(\frac{r}{a}\sqrt{u^2 - x^2}), & r > b, \\ \frac{B}{Y_0(x)}\left[J_0(\frac{r}{a}x)Y_0(x) - Y_0(\frac{r}{a}x)J_0(x)\right], & a < r < b \end{cases} \tag{5}$$

Here,

$$\gamma = \frac{\beta}{k_0},\ x = k_0 a\sqrt{\epsilon - \gamma^2},\ u = k_0 a\sqrt{\epsilon - 1},\ s = \frac{b}{a} > 1, \tag{6}$$

and $J_k(x)$, $Y_k(x)$, $K_0(x)$, and $K_1(x)$ denote, respectively, the Bessel and Neumann functions of the order $k = 0, 1$ and the McDonald functions.

3. ANALYSIS

Applying to (5) the transmission (continuity) condition we obtain the dispersion equation (DE) for surface waves

$$F_G(x,u,s) \equiv G_g(x,u,s) - F_g(x,s) = 0. \tag{7}$$

where

$$G_g(x,u,s) = \epsilon\sqrt{u^2 - x^2}\frac{K_0(s\sqrt{u^2 - x^2})}{K_1(s\sqrt{u^2 - x^2})}, \qquad F_g(x,s) = x\frac{\Phi_1(x,s)}{\Phi_2(x,s)}, \tag{8}$$

$$\Phi_1(x,s) = J_0(sx)Y_0(x) - J_0(x)Y_0(sx), \qquad \Phi_2(x,s) = J_0(x)Y_1(sx) - J_1(sx)Y_0(x).$$

Denote by $h_m^0(s)$, and $h_m^{10}(s)$ $(m = 1, 2, \dots)$ zeros of Φ_1 and Φ_2. For a fixed $s > 1$, all these zeros are real and simple points [2]; the sequences $\{h_m^0(s)\}$ and $\{h_m^{10}(s)\}$ are increasing w.r.t index m and alternate on the positive semi-axis. $h_m^{10}(s)$ are decreasing functions of $s > 1$. The domain of function $F_G(x, u, s)$ is

$$D_{F_G}(u,s) : \{x : |\, x \,| \le u \setminus H_G(u,s), \quad u > 0, \quad s \ge 1, \}$$
$$H_G(u,s) = \left\{h_m^{01}(s) : 0 \le h_m^{01}(s) \le u,\ m = 1, 2, \dots N(u)\right\},$$
$$H_G(u,s) = \emptyset\ (N(u) = 0) \quad \text{if all } h_m^{01}(s) > u \quad (m = 1, 2, \dots N(u)).$$

According to the conditions on parameters we will consider $F_G(x, u, s)$ on the set

$$D^+_{F_G}(u,s) : \{(x,u,s) : 0 \le x \le u \setminus H_G(u,s),\ u > 0,\ s > 1\}.$$

Figure 2 illustrates the proof of existence of roots of DE (7).

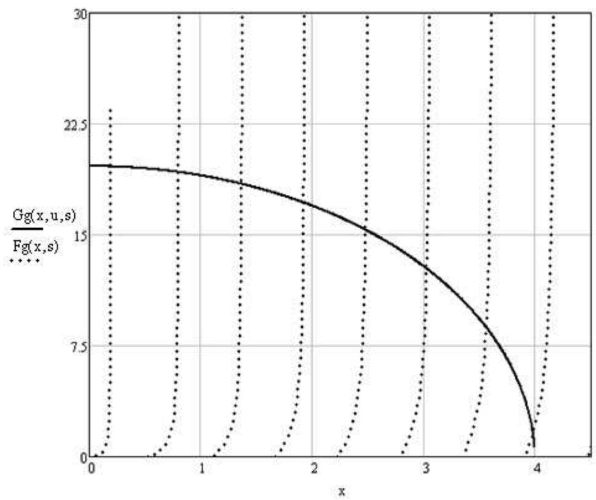

Figure 2: Behavior of functions $G_g(x; u)$ and $F_g(x)$ (8) for the Goubau line ($\epsilon = 5$) for $s = 6.615$.

Perform the analysis of higher-order surface waves using parameter-differentiation method and reduction to the solution of the Cauchy initial value problems. Note first that (i) for every $u > 0$ there exists a (minimal positive) zero $h_m^0(s)$ of function $\Phi_1(x, s)$ such that

$$u = h_m^0(s_m^0) \quad \text{for one and only one} \quad s = s_m^0 = s(u) > 1, \quad m = 1, 2, \dots; \tag{9}$$

(ii) according to the explicit form (7) of the DE $F_0(x = u, u, s_m^0) = 0$ for any $u > 0$ and $\frac{\partial F_0(u,u,s)}{\partial x} \neq 0$. Numbers s_m^0 can be determined, for every given $u > 0$, by solving numerically the equation $\Phi_1(u, s) = 0$. Thus $x = x^m(s) = x^m(s)$ can be determined as a regular perturbation of the value $x = h_m^0$ attained at $s = s_m^0$: for a fixed $u > 0$, find the implicit function $x^m = x^m(s)$ as a (real) solution to the Cauchy problem

$$\begin{cases} \frac{dx^m}{ds} = \tilde{F}_0(x,u,s), \quad s > s_m^0, \\ x^m(s_m^0) = h_m^0(s_m^0),\ m = 1, 2, \dots, N_0(u). \end{cases} \tag{10}$$

Note that $x^m(s)$ satisfies the condition

$$h_m^0(s) > x^m(s) > h_m^{01}(s), \quad m = 1, 2, \dots, N_0(u). \tag{11}$$

We prove that if a given $u = k_0 a\sqrt{\epsilon - 1}$ satisfies the condition

$$u = k_0 a\sqrt{\epsilon - 1} > h_m^0(s),$$

the Cauchy problem (10) is uniquely solvable. Thus, for arbitrary $s = b/a > 1$ that may be sufficiently close to unity, there exists a root $x^1 = x^1(s)$ of DE (7) [more exactly, implicit function

$x^1(s)$ specified by Equation (7) in the vicinity of point $A_{0]}^1$. In other words, for an arbitrarily thin dielectric layer there exists (at least one) surface wave propagating in the Goubau line under the condition

$$u = k_0 a\sqrt{\epsilon - 1} \geq h_1^0(s). \tag{12}$$

Relationship (12) is the necessary condition that provides the existence of at least one such wave. Taking into account the domain of function $F_0(x, u, s)$ and $\tilde{F}_0(x, u, s)$ we have the condition $x \in (0, k_0 a\sqrt{\epsilon})$.

4. STRUCTURE OF HIGHER-ORDER WAVES

The cross-sectional structure of the nth higher-order wave is characterized by the number of zeros (oscillations) of potential function $\phi_n(r)$ on (a, b); namely, $\phi_n(r)$ has exactly n zeros (see Fig. 3) and the intervals between two neighboring zeros contained in the interval tend to zero. We calculate normalized propagation constants γ of the fundamental (with no oscillations) and higher-order surface modes v.s. parameter s for different values of parameter u using numerical solution of (10). Cutoff values of s are determined by condition (2). The results are summarized in Fig. 4. The potential functions ϕ_n preserve their types (number n of zeros or oscillations) along the dispersion curves.

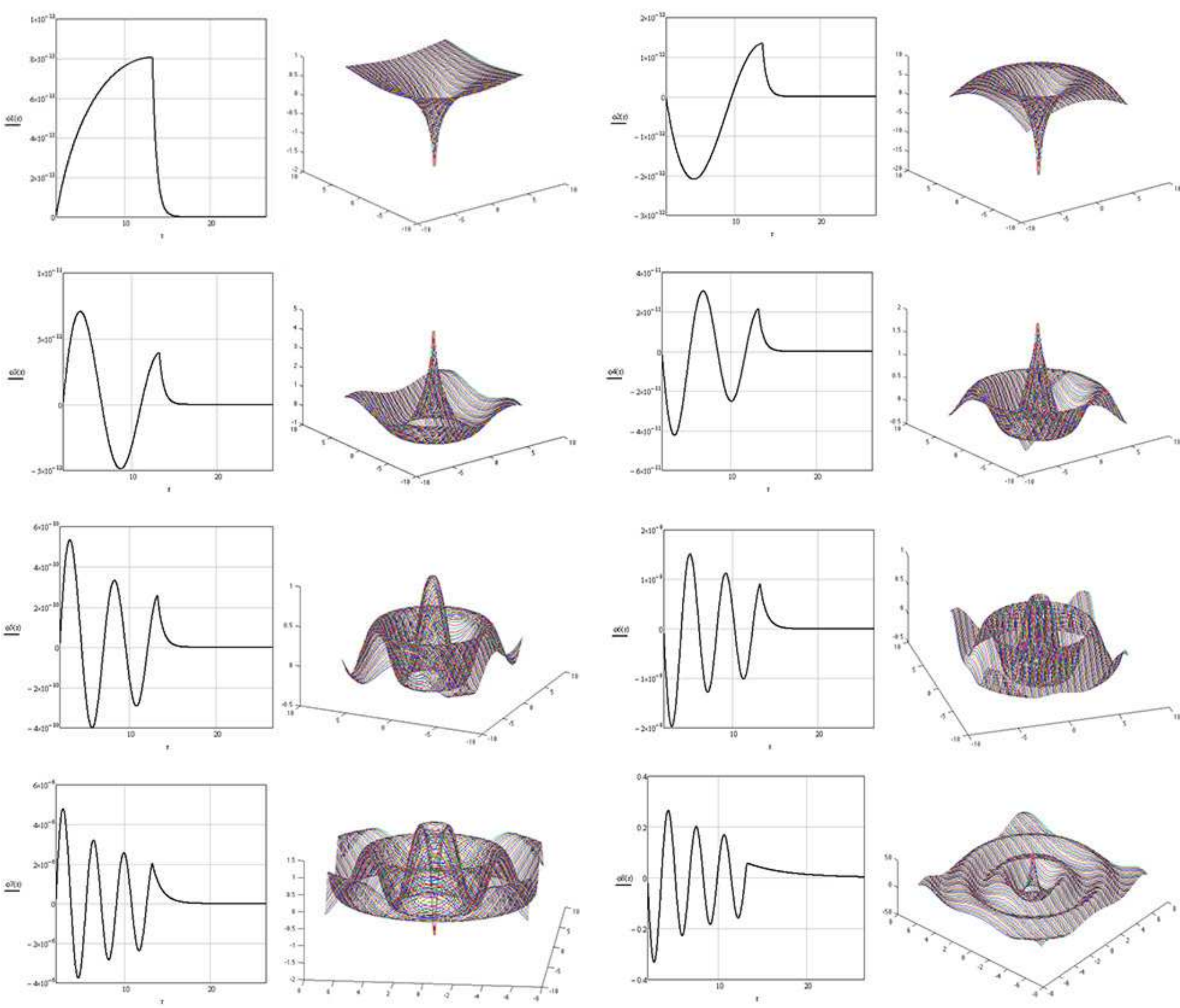

Figure 3: Potential function $\phi_n(r)$ (5) of the higher-order surface waves of the Goubau line calculated for different eigenvalue indices $n = 0, 1, \ldots, 7$ at $a = 2$, $b = 13.23$, and $\epsilon = 5$.

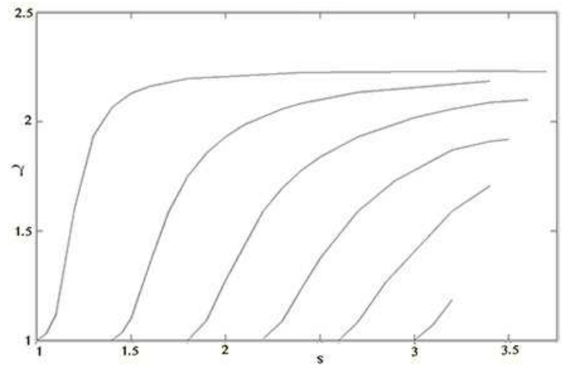

Figure 4: Normalized propagation constants γ (vertical axis) of the fundamental (with no oscillations) and higher-order surface modes of the Goubau line v.s. parameter s calculated for value $u = 8$.

5. CONCLUSION

We have shown that higher-order surface modes in the GL exist for arbitrarily thin dielectric coating if the electrical cross-sectional dimension is sufficiently large. We have determined the cutoff values and cross-sectional structure of the higher-order surface modes in the coating and have demonstrated that their spectrum can be efficiently calculated from the numerical solution of Cauchy initial value problems obtained using parameter differentiation. We have reduced the analysis of the spectrum of symmetric surface modes to singular and regular Sturm-Liouville boundary eigenvalue problems on the half-line and on the intervals. We have shown that the spectrum of surface waves of the Goubau line with an infinitely thin dielectric coating is a perturbed set of zeros of a well-defined function. The established properties of surface waves enable one to use GLs as near-field testing and sensor devices and for cloaking [7, 8].

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