Projection Method for Solving Scalar Problem of Diffraction of a Plane Wave on a System of Two- and Three-dimensional Obstacles

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Abstract— We investigate scalar problem of diffraction of a plane wave on a system of obstacles consisting of disjoint smooth screens Ω_i and volume inhomogeneous bodies Q_j. The original boundary value problem leads to a system of weakly singular integral equations on two- and three-dimensional manifolds. We obtain important results on smoothness of the solution to the system of the integral equations in the interior points of the screens, on the equivalence of the integral equations to the original boundary value problem. Finally we prove the invertibility of the integral operator. We propose Galerkin method for numerical solving of the integral equations. We prove the approximation property for the piecewise constant basis functions as well as the statement of Galerkin method convergence.

1. INTRODUCTION

The scalar problems of diffraction by two- or three-dimensional obstacles only are well known.

For example, in monograph [1] and papers [2–4] one can find detailed description and investigation of acoustic diffraction by volume obstacles. In [5], integral equation of Lippmann-Schwinger type is investigated. The problem of diffraction by acoustically soft (or hard) screens is investigated in [6] — Here the boundary value problem is reduced do integral equation which is treated as pseudodifferential equation in Sobolev spaces.

In this paper, we investigate new problem of diffraction by a compound scatterer consisting of both two- and three-dimensional obstacles. Following [6] we reduce boundary value problem for Helmholtz equation to weakly singular integral equations and prove the invertibility of the integral operator. For numerical solving of the problem we propose Galerkin method.

2. STATEMENT OF THE PROBLEM

Let $\Omega = \bigcup\limits_j \Omega_j$, $(\bar{\Omega}_{j_1} \bigcap \bar{\Omega}_{j_2} = \varnothing,\ j_1 \neq j_2)$ be a union of oriented screens (two-dimensional C^∞-manifolds) with C^∞-smooth boundary $\partial\Omega = \bar{\Omega} \setminus \Omega$. We define δ-neighbourhood of $\partial\Omega_\delta$ as follows: $\partial\Omega_\delta = \{x \in \mathbb{R}^3 : dist(x, \partial\Omega) < \delta\}$.

Let $Q = \bigcup\limits_i Q_i$, $(\bar{Q}_{i_1} \bigcap \bar{Q}_{i_2} = \varnothing,\ i_1 \neq i_2)$ be a union of bounded volume bodies with piecewise smooth boundary ∂Q. Assume also that $\bar{Q} \cap \bar{\Omega} = \varnothing$. The body Q is inhomogeneous

$$k(x) = \begin{cases} k_e, & x \in \left(\bar{Q} \cup \bar{\Omega}\right)^c \\ k_i(x), & x \in \bar{Q}_i, \end{cases}$$

where $k_i(x) \in C^\infty(\bar{Q}_i)$ and $M^c = \mathbb{R}^3 \setminus M$. The media outside obstacles is homogeneous – $k_e =$ const. The conditions $\Re k(x) > 0$ and $\Im k(x) \geqslant 0$ hold everywhere in $\bar{\Omega}^c$. The incident field is $u_0(x) = e^{ik_e(\alpha x_1 + \beta x_2 + \gamma x_3)}$, $x \in \mathbb{R}^3$.

We find the solution

$$u \in C^2\left((\partial Q \cup \bar{\Omega})^c\right) \bigcap C^1\left(\bar{Q}^c \setminus \bar{\Omega}\right) \bigcap C^1(\bar{Q}) \bigcap_{\delta > 0} C((\partial\Omega_\delta)^c), \tag{1}$$

to the Helmholtz equation

$$\Delta u(x) + k^2(x) u = 0, \quad x \in \left(\partial Q \cup \bar{\Omega}\right)^c, \tag{2}$$

which satisfies transmission conditions, Dirichlet condition on the screen

$$[u]|_{\partial Q} = 0, \quad \left.\left[\frac{\partial u}{\partial \mathbf{n}}\right]\right|_{\partial Q} = 0, \quad u\,|_\Omega = 0, \tag{3}$$

as well as the condition of finite energy [6]

$$u \in H^1_{loc}\left(\mathbb{R}^3\right). \tag{4}$$

The scattered field $u_s = u - u_0$ should satisfy Sommerfeld conditions at the infinity:

$$\frac{\partial u_s}{\partial r} = ik_e u_s + o\left(r^{-1}\right),\ (\Im k_e = 0);\ u_s(r) = O\left(r^{-2}\right),\ (\Im k_e > 0);\ r := |x| \to \infty. \tag{5}$$

Definition. Solution $u(x)$ to (2)–(5) is called *quasiclassical* if it satisfies (1) (compare with [7]).
Statement 1. [9] For $\Im k(x) \geq 0$ there is at most one solution to (2)–(5).

3. THE SYSTEM OF INTEGRAL EQUATIONS

We represent the solution to the problem via composition

$$u = u_0 + u_1 + u_2, \tag{6}$$

where u_0 is the incident field; u_1 is the field scattered by the screens only:

$$u_1(x) = \int\limits_{\Omega} G(x,y)\varphi(y)ds_y,$$

here $G(x,y) = \frac{1}{4\pi} \cdot \frac{e^{ik_e|x-y|}}{|x-y|}$, and φ is unknown.

Using methods of potential theory, taking (6) and Dirichlet conditions into account we obtain the following system of integral equations:

$$\begin{cases} (I-A)u + K_1\varphi = u_0|_Q \\ K_2 u + S\varphi = u_0|_\Omega \end{cases}. \tag{7}$$

Here

$$Au = \int\limits_Q \left(k^2(y) - k_e^2\right) G(x,y)u(y)dy, \qquad K_1\varphi = -\int\limits_\Omega G(x,y)\varphi(y)ds_y,\ x \in Q;$$

$$K_2 u = -\int\limits_Q \left(k^2(y) - k_e^2\right) G(x,y)u(y)dy, \quad S\varphi = -\int\limits_\Omega G(x,y)\varphi(y)ds_y,\ x \in \Omega.$$

Let us define matrix operator of the system (7):

$$\hat{L} = \hat{L}_1 + \hat{L}_2 = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} + \begin{pmatrix} -A & K_1 \\ K_2 & 0 \end{pmatrix} \tag{8}$$

and treat it as a pseudodifferential operator in Sobolev spaces (see [8]):

$$\hat{L} : L_2(Q) \times \tilde{H}^{-1/2}\left(\bar{\Omega}\right) \to L_2(Q) \times H^{1/2}\left(\Omega\right). \tag{9}$$

As the operator S is a Fredholm with zero index [6] we can prove

Statement 2. [9] $\hat{L}$ is a Fredholm operator with zero index.

For the solution of (7) the following statements hold (see [9]):

Statement 3. If $u_0 \in C^\infty(\mathbb{R}^3)$ and pair $(u, \varphi) \in L_2(Q) \times \tilde{H}^{-1/2}(\bar{\Omega})$ is a solution to the system (7) then $\varphi \in C^\infty(\Omega)$, and u satisfies (1).

Statement 4. Let $u_0 \in C^\infty(\mathbb{R}^3)$. Then the BVP (2)–(5) is equivalent to (7).

Taking into account the aforementioned statements we obtain the main result on the solvability of the problem under consideration:

Statement 5. The matrix operator $\hat{L}$ is continuously invertible.

4. GALERKIN METHOD

We consider Galerkin method for equation $Af = g$, where A is invertible operator that maps Hilbert space H_1 onto its anti-dual H_2. The approximate solution is sought as follows: $f_N = \sum\limits_{i=1}^{N} c^i w_i$; basis functions w_i satisfy the *approximation condition*:

$$\forall f \in H_1 \quad \lim_{N\to\infty} \inf_{f_N \in H_{1,N}} \|f - f_N\| = 0, \tag{10}$$

the coefficients c^i satisfy the system of linear equations

$$\sum_{j=1}^{N} c^j (Aw_j, w_i) = (g, w_i), \quad i = 1, \dots, N; \tag{11}$$

here $(\cdot, \cdot)$ denotes natural antiduality between H_1 and H_2.

Let us define the approximate solution $f_N = (u_n, \varphi_m)$ to the system (7) as follows:

$$u_n(x) = \sum_{i=1}^{n} c_u^i v_i(x),\ x \in Q; \qquad \varphi_m(x) = \sum_{i=1}^{m} c_\varphi^i \psi_i(x),\ x \in \Omega.$$

Here $v_i(x)$, $\psi_i(x)$ are basis functions. First, we define functions v_i in Q.

Let $\Pi = \{x : a_1 < x_1 < a_2,\ b_1 < x_2 < b_2,\ c_1 < x_3 < c_2\}$ be a parallelepiped: $Q \subset \Pi$. Then

$$v_i(x) = v_{i_1 i_2 i_3}(x) := \begin{cases} 1, & x \in Q \cap \bar{\Pi}_{i_1 i_2 i_3} \\ 0, & x \notin Q \cap \bar{\Pi}_{i_1 i_2 i_3} \end{cases} \tag{12}$$

where $\Pi_{i_1 i_2 i_3} = \{x : x_{1,i_1} < x_1 < x_{1,i_1+1},\ x_{2,i_2} < x_2 < x_{2,i_2+1},\ x_{3,i_3} < x_3 < x_{3,-i_3}\}$, $x_{1,i_1} = a_1 + \frac{a_2-a_1}{n_1} i_1$, $x_{2,i_2} = b_1 + \frac{b_2-b_1}{n_2} i_2$, $x_{3,i_3} = c_1 + \frac{c_2-c_1}{n_3} i_3$; $i_k = 0, \dots, n_k - 1$, $n = n_1 n_2 n_3$.

Let Ω be a *plane* screen; then basis functions $\psi_i = \psi_{i_1, i_2}$ $(i_k = 0, \dots, m_k - 1)$ can be defined in the same way.

Let Ω be a *not plane* screen with parametrization $x = x(t_1, t_2)$, $(t_1, t_2) \in D \subset \mathbb{R}^2$, where D – is a compact (e.g., a rectangle). Then $\psi_i = \psi_{i_1 i_2}$ are finite functions with support $supp(\psi_i) = x(\Pi_{i_1, i_2}) \subset \Omega$. If $\lim\limits_{m\to\infty} \max\limits_{i_1, i_2} \mu_D(\Pi_{i_1, i_2}) = 0$; then owing to smoothness of the screen we obtain $\lim\limits_{m\to\infty} \max\limits_{i} \mu_\Omega(supp(\psi_i)) = 0$.

Statement 6. [9] The approximation condition holds for basis functions (v_i, ψ_j) in $H_1 = L_2(Q) \times \tilde{H}^{-1/2}(\bar{\Omega})$.

As S is a coercive operator (see. [6], p.245) the following statement holds

Statement 7. Galerkin method (11) is convergent for $\hat{L}$.

Galerkin method leads to a system of linear equations $[\hat{L}][\mathbf{u}] = [\mathbf{u_0}]$; the augmented matrix can be represented in block form as follows:

$$\left[\hat{L}\right] = \left[\begin{array}{c|c|c} [A] & [K_1] & [u_{0,1}] \\ \hline [K_2] & [S] & [u_{0,2}] \end{array}\right] \tag{13}$$

where

$$[A]_{ij} = vol(\Pi_i)\delta_{ij} - \int\limits_{\Pi_i}\int\limits_{\Pi_j} (k^2(y) - k_e^2) G(x,y) dy dx$$

$$[K_1]_{ij} = -\int\limits_{\Pi_i}\int\limits_{\omega_j} G(x,y) ds_y dx, \quad [K_1]_{ij} = -\int\limits_{\omega_i}\int\limits_{\Pi_j} \left(k^2(y) - k_e^2\right) G(x,y) dy ds_x,$$

$$[S]_{ij} = -\int\limits_{\omega_i}\int\limits_{\omega_j} G(x,y) ds_y ds_x, \quad [u_{0,1}]_i = \int\limits_{\Pi_i} u_0(x) dx, \quad [u_{0,2}]_i = \int\limits_{\omega_i} u_0(x) ds_x.$$

To obtain good precision of the approximate solution we must use a sufficient number of basis functions — at least 10 per wavelength for each dimension of the scatterers. For this reason we get dense matrices $[A]$ of order $N \sim 10^3$ and more. As the kernels of integral operators depend on $|x-y|$, the matrices $[A]$ under consideration are block-Toeplitz. This allows to accelerate Galerkin procedure and perform calculations in reasonable time.

We represent below some results of numerical tests.

Consider a cube $Q = \{x \in \mathbb{R}^3 : x_i \in [0; 0.5]\}$ and a rectangle screen $\Omega = \{x \in \mathbb{R}^3 : x_1, x_2 \in [0; 0.5],\ x_3 = 1\}$; let $k_e = 2\pi$, and $u_0 = e^{ik_e x_3}$; $n_i = m_k = 12$.

Figure 1 represents the absolute values $|\varphi(x)|$ of the solution $\varphi(x)$ in Ω and $|u(x)|$ in three cross-sections of the cube (given $x_3 = 0.5$, $x_3 = 0.25$, $x_3 = 0$.)

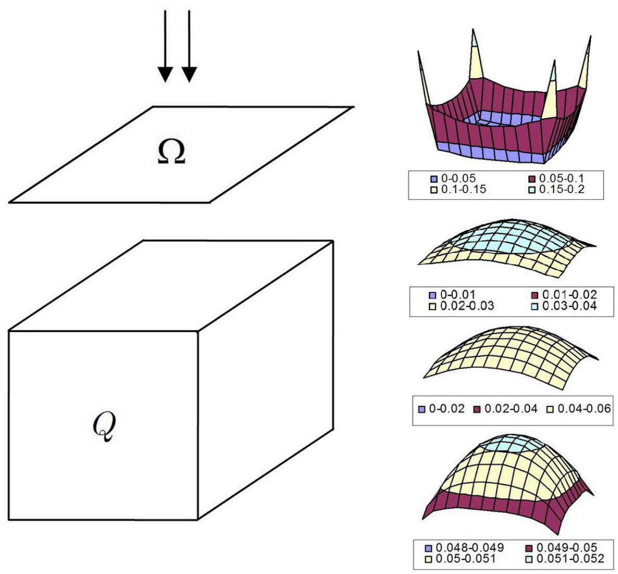

Figure 1.

Excluding blocks K_i (and $[A]$) allows to solve simultaneously the problems of diffraction on a volume body and a screen (on the screen only).

5. CONCLUSION

The scalar problem of diffraction of a plane wave on a system of bodies and screens is investigated. The system of integral equations is obtained and is proved to be uniquely solvable. The convergence of Galerkin method is proved, several numerical results are presented.

ACKNOWLEDGMENT

This work was supported by the Russian Science Foundation (grant 14-11-00344).

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