Propagation of Electromagnetic Waves along a Nonlinear Inhomogeneous Cylindrical Waveguide

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Abstract— Electromagnetic TE waves propagating in an inhomogeneous nonlinear cylindrical waveguide are considered. The permittivity inside the waveguide is described by the Kerr law. Inhomogeneity of the waveguide is modeled by a nonconstant term in the Kerr law. Physical problem is reduced to a nonlinear eigenvalue problem for an ordinary differential equation. Theorem of existence of propagation constants and their localization is formulated. Conditions of k waves existence are found.

1. INTRODUCTION

We investigate guided waves in a nonlinear dielectric inhomogeneous cylindrical waveguide filled with Kerr medium. The waveguide is placed in cylindrical coordinate system $O\rho\varphi z$, where axis z coincides with axis of the waveguide. The permittivity inside the waveguide is $\varepsilon = \varepsilon_2(\rho) + \alpha|\mathbf{E}|^2$, where $\varepsilon_2(\rho)$ is the inhomogeneity, $\alpha > 0$ is a real constant, and $\mathbf{E}$ is the complex amplitude of an electromagnetic field.

This problem and similar ones lead to nonlinear transmission eigenvalue problems for ordinary differential equations (a short review of the problems and results see in [1]). Eigenvalues in these problems correspond to propagation constants of the waveguides. In these problems differential equations depend nonlinearly either on sought-for functions and the spectral parameter. The transmission conditions depend nonlinearly on the spectral parameter. The main goal is to prove existence of eigenvalues and determine their localization. Existence and localization can be derived from the dispersion equation (DE). The DE is an equation with respect to the spectral parameter. There are two ways to obtain the DE. The first one is to integrate the differential equations and obtain, using the transmission conditions, the DE. This way is of very limited applicability, as it is very rarely possible to find explicit solutions of nonlinear differential equations. The second one is a very general approach based on a reduction of the differential equations to integral equations using the Green function. Here we consider this very method. In spite of the fact that by this method the DE is found in an implicit form, it is possible to prove existence of eigenvalues and find their localization.

Two circumstances are important for the following analysis. First, in the case of a homogeneous waveguide ($\varepsilon_2(\rho) \equiv \mathrm{const}$) this Green function can be found explicitly. Second, the DE of the nonlinear homogeneous case can be written as $\mathrm{DE}_{\mathrm{lin}} + \mathrm{T}_{\mathrm{nonlin}} = 0$, where $\mathrm{DE}_{\mathrm{lin}}$ is a linear problem term and $\mathrm{T}_{\mathrm{nonlin}}$ is an extra nonlinear term. Here the linear problem term is written in an explicit form. Moreover, the equation $\mathrm{DE}_{\mathrm{lin}} = 0$ is well known and examined. Its roots are also known. All this allows proving the existence of the nonlinear problem solutions at least near to the linear problem solutions. The nonconstant term $\varepsilon_2(\rho)$ dramatically changes the situation. In this case we cannot find the necessary Green function explicitly, so we investigate it in an implicit form. The DE of the nonlinear inhomogeneous case can also be written as $\mathrm{DE}_{\mathrm{lin}} + \mathrm{T}_{\mathrm{nonlin}} = 0$. However, in this case the term $\mathrm{DE}_{\mathrm{lin}}$ is found in an implicit form and roots of the equation $\mathrm{DE}_{\mathrm{lin}} = 0$ are unknown. In spite of the fact that the method here looks similar to the method in [2, 3] we solve a radically different problem.

2. STATEMENT OF THE PROBLEM

Let us consider three-dimensional space $\mathbb{R}^3$ with cylindrical coordinate system $O\rho\varphi z$. The space is filled by isotropic medium with constant permittivity $\varepsilon_1 \geqslant \varepsilon_0 > 0$, where ε_0 is the permittivity of free space. In this medium a cylindrical waveguide is placed. The waveguide is filled by isotropic nonmagnetic medium and has cross section $W := \{(\rho,\varphi) : \rho \leqslant R, 0 \leqslant \varphi < 2\pi\}$ and its generating line (the waveguide axis) is parallel to the axis Oz. We shall consider electromagnetic waves propagating along the waveguide axis. Everywhere below $\mu = \mu_0$ is the permeability of free space.

We use Maxwell's equations in the following form [4]

$$\mathrm{rot}\,\tilde{\mathbf{H}} = \partial_t \tilde{\mathbf{D}}, \quad \mathrm{rot}\,\tilde{\mathbf{E}} = -\partial_t \tilde{\mathbf{B}}, \tag{1}$$

where $\tilde{\mathbf{D}} = \varepsilon\tilde{\mathbf{E}}$, $\tilde{\mathbf{B}} = \mu\tilde{\mathbf{H}}$ and $\partial_t = \partial/\partial t$. Field $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ is the total field.

Real monochromatic field $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ in the medium can be written in the following form

$$\begin{aligned} \tilde{\mathbf{E}}(\rho, \varphi, z, t) &= \mathbf{E}^{+}(\rho, \varphi, z)\cos\omega t + \mathbf{E}^{-}(\rho, \varphi, z)\sin\omega t; \\ \tilde{\mathbf{H}}(\rho, \varphi, z, t) &= \mathbf{H}^{+}(\rho, \varphi, z)\cos\omega t + \mathbf{H}^{-}(\rho, \varphi, z)\sin\omega t, \end{aligned} \tag{2}$$

where ω is the circular frequency; $\mathbf{E}^{+}$, $\mathbf{E}^{-}$, $\mathbf{H}^{+}$, $\mathbf{H}^{-}$ are real required vectors.

Let us form complex amplitudes $\mathbf{E}$, $\mathbf{H}$:

$$\mathbf{E} = \mathbf{E}^{+} + i\mathbf{E}^{-}, \quad \mathbf{H} = \mathbf{H}^{+} + i\mathbf{H}^{-}. \tag{3}$$

It is clear that $\tilde{\mathbf{E}} = \mathrm{Re}\{\mathbf{E}e^{-i\omega t}\}$ and $\tilde{\mathbf{H}} = \mathrm{Re}\{\mathbf{H}e^{-i\omega t}\}$, where $\mathbf{E} = (E_\rho, E_\varphi, E_z)^T$ and $\mathbf{H} = (H_\rho, H_\varphi, H_z)^T$, and the components depend on three spatial variables.

It is known (see, e.g., [5–7]) that the Kerr law in isotropic medium for a monochromatic wave $\mathbf{E}e^{-i\omega t}$ has the form $\varepsilon = \varepsilon_2 + a|\mathbf{E}|^2$, where $\mathbf{E}$ is complex amplitude, a is the coefficient of the nonlinearity. Thus, for a monochromatic wave $(\mathbf{E}, \mathbf{H})e^{-i\omega t}$ the complex amplitudes satisfy stationary Maxwell's equations

$$\mathrm{rot}\,\mathbf{E} = i\omega\mu\mathbf{H}, \quad \mathrm{rot}\,\mathbf{H} = -i\omega\varepsilon\mathbf{E}, \tag{4}$$

the continuity condition for the tangential components on the media interfaces (on the boundary of the waveguide) and the radiation condition at infinity: the electromagnetic field decays as $O(\rho^{-1})$ when $\rho \to \infty$.

Inside the waveguide the permittivity has the form $\varepsilon = \varepsilon_2(\rho) + a\,|\mathbf{E}|^2$, where where a is a real positive value, $\varepsilon_2(\rho)$ is a real continuous function. Here $\varepsilon_2(\rho)$ is a linear part of the permittivity.

Thereby, passing from time dependent Equation (1) to time independent Equation (4) is grounded on the previous consideration.

The solutions to Maxwell's equations are sought in the entire space.

Geometry of the problem is shown in Fig. 1. The waveguide is infinite along axis Oz.

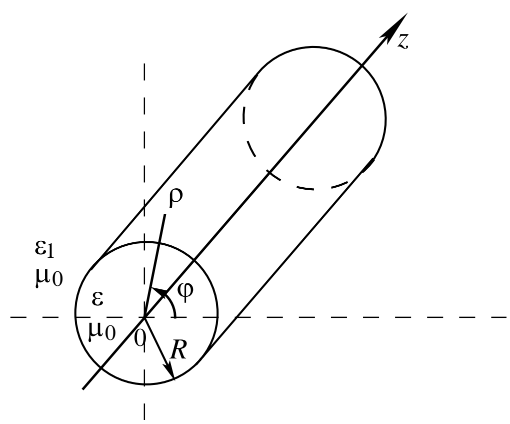

Figure 1: Geometry of the problem.

Consider the TE waves with harmonical dependence on time

$$\mathbf{E}e^{-i\omega t} = e^{-i\omega t}(0, E_\varphi, 0)^T, \quad \mathbf{H}e^{-i\omega t} = e^{-i\omega t}(H_\rho, 0, H_z)^T,$$

where $\mathbf{E}$, $\mathbf{H}$ are the complex amplitudes.

Substituting the complex amplitudes into Maxwell's Equation (4), we obtain that they do not depend on φ.

Waves propagating along waveguide axis Oz depend harmonically on z. This means that the fields components have the form

$$E_\varphi = \mathrm{E}_\varphi(\rho)e^{i\gamma z}, \quad H_\rho = \mathrm{H}_\rho(\rho)e^{i\gamma z}, \quad H_z = \mathrm{H}_z(\rho)e^{i\gamma z},$$

where γ is the unknown real spectral parameter of the problem (propagation constant). Substituting these components into Maxwell's Equation (4) and introducing notation $u(\rho) := \mathrm{E}_\varphi(\rho)$, $k_0^2 = \omega^2\mu\varepsilon_0$ we obtain

$$\rho u'' + u' - \rho^{-1}u + \rho\left(k_0^2\tilde{\varepsilon} - \gamma^2\right)u = 0, \tag{5}$$

where $\tilde{\varepsilon} = \begin{cases} \varepsilon_1, & \rho > R, \\ \varepsilon_2(\rho) + au^2, & \rho \leqslant R. \end{cases}$

Also we assume that function u is sufficiently smooth

$$u(\rho) \in C[0, +\infty) \cap C^1[0, +\infty) \cap C^2(0, R) \cap C^2(R, +\infty).$$

We will seek γ under conditions $k_0^2\varepsilon_1 < \gamma^2$.

In the domain $\rho > R$ we have $\tilde{\varepsilon} = \varepsilon_1$. From (5) we obtain the equation

$$u'' + \rho^{-1}u' - \rho^{-2}u + k_1^2 u = 0, \tag{6}$$

where $k_1^2 = k_0^2\varepsilon_1 - \gamma^2$. It is the Bessel equation. In accordance with the radiation condition its solution is

$$u(\rho) = \tilde{b}K_1(|k_1|\rho), \quad \rho > R, \tag{7}$$

where $K_1(z)$ is the Macdonald function, $\tilde{b}$ is a constant of integration. The radiation condition is fulfilled since $K_1(|k_1|\rho) \to 0$ as $\rho \to \infty$.

In the domain $\rho < R$ we have $\tilde{\varepsilon} = \varepsilon_2(\rho) + au^2$. From (5), we obtain the equation

$$(\rho u')' + \left(k^2(\rho)\rho - \rho^{-1}\right) u + \alpha\rho u^3 = 0, \tag{8}$$

where $k^2(\rho) = k_2^2(\rho) - \gamma^2$, $k_2^2(\rho) = k_0^2\varepsilon_2(\rho)$, and $\alpha = ak_0^2$.

Tangential components of electromagnetic field are known to be continuous at media interfaces. Hence, we obtain $E_\varphi(R + 0) = E_\varphi(R - 0)$, $H_z(R + 0) = H_z(R - 0)$. These conditions imply the transmission conditions for the functions $u(\rho)$ and $u'(\rho)$

$$[u]|_{\rho=R} = 0, \ [u']\big|_{\rho=R} = 0, \tag{9}$$

where $[f]|_{x=x_0} = \lim\limits_{x\to x_0-0} f(x) - \lim\limits_{x\to x_0+0} f(x)$.

The problem P is to prove existence of eigenvalues γ for which a nontrivial solution $u(\rho;\gamma)$ to (6), (8), which is called an eigenfunctions, exists; this solution must satisfy transmission conditions (9) and the radiation condition at infinity: each eigenfunction decay as $O(\rho^{-1})$ when $\rho \to \infty$.

3. NONLINEAR INTEGRAL EQUATION AND DISPERSION EQUATION

Suppose that the Green function $G_k(\rho, \rho_0; \lambda)$ exists for the following boundary value problem

$$L_k G_k = -\delta(\rho - \rho_0), \quad G\,|_{\rho=0} = G'\,|_{\rho=R} = 0 \quad (0 < \rho_0 < R)\,,$$

where $L_k := \frac{d}{d\rho}(\rho\frac{d}{d\rho}) + (k^2(\rho)\rho - \frac{1}{\rho})$; here we place index k in order to stress that the operator and the Green function depend on $k(\rho)$.

In this case the Green function has the representation (see, for example, [8])

$$G_k(\rho, \rho_0; \lambda) = -\frac{v_i(\rho)v_i(\rho_0)}{\lambda - \lambda_i} + G_1(\rho, \rho_0; \lambda) \tag{10}$$

in the vicinity of eigenvalue λ_i. Here $\lambda := \gamma^2$ and $G_1(\rho, \rho_0; \lambda)$ is regular w.r.t. λ in the vicinity of λ_i; λ_n, $v_n(\rho)$ are complete (real) eigenvalues and orthonormal eigenfunctions systems of the boundary eigenvalue problem

$$(\rho v_n')' + \left(k_2^2(\rho)\rho - \rho^{-1}\right) v_n = \lambda_n \rho v_n; \quad v_n\,|_{\rho=0} = v_n'\,|_{\rho=R} = 0\,. \tag{11}$$

The Green function exists if $\lambda \neq \lambda_i$.

For $\varepsilon_2 \equiv \text{const}$ explicit form of the Green function is given in [2].

Let us write Equation (8) in the operator form

$$L_k u + \alpha B(u) = 0, \quad B(u) = \rho u^3(\rho). \tag{12}$$

Using the second Green formula [9] and Equation (12) we obtain the nonlinear integral representation of solution $u(\rho_0)$ of Equation (8) on the segment $[0, R]$

$$u(\rho_0) = \alpha \int\limits_0^R G_k(\rho, \rho_0)\rho u^3(\rho)d\rho + Ru'(R - 0)G_k(R, \rho_0), \quad 0 \leqslant \rho_0 \leqslant R. \tag{13}$$

Using the transmission condition $u'(R-0) = u'(R+0)$ we can rewrite Equation (13)

$$u(\rho_0) = \alpha \int_0^R G_k(\rho, \rho_0)\rho u^3(\rho)d\rho + Ru'(R+0)G_k(R, \rho_0), \quad 0 \leqslant \rho_0 \leqslant R. \tag{14}$$

Using Equation (14) and the transmission condition $u(R-0) = u(R+0)$ we obtain the DE w.r.t. the propagation constant

$$u(R+0) = \alpha \int_0^R G_k(\rho, R)\rho u^3(\rho)d\rho + Ru'(R+0)G(R, R). \tag{15}$$

4. EXISTENCE AND LOCALIZATION OF EIGENVALUES

Taking into account formula (7), DE (15) can be represented in the form

$$K_1(|k_1|R) - |k_1|RK_1'(|k_1|R)\,G_k(R, R; \lambda) = \frac{\alpha}{\tilde{b}} \int_0^R G_k(\rho, R; \lambda)\rho u^3(\rho)d\rho. \tag{16}$$

It is clear that DE (16) depends on $\tilde{b}$. Here $\tilde{b}$ is an initial condition. This is a peculiarity of nonlinear eigenvalue problems. For the linear problem (if $\alpha = 0$) we obtain, as it is expected, the DE, which does not depend on the initial condition.

DE (16) can be rewritten in the following form

$$g(\lambda) = \alpha F(\lambda),$$

where

$$g(\lambda) = K_1(|k_1|R) + (|k_1|RK_0(|k_1|R) + K_1(|k_1|R))\,G_k(R, R; \lambda),$$

and

$$F(\lambda) = \int_0^R G_k(\rho, R; \lambda)\rho u^3(\rho)d\rho.$$

The zeros of the function $\Phi(\lambda) \equiv g(\lambda) - \alpha F(\lambda)$ are eigenvalues of the problem P.

At first, consider the linear problem $g(\lambda) = 0$. This equation can be rewritten in the form

$$G_k(R, R; \lambda) = -\frac{K_1(|k_1|R)}{|k_1|RK_0(|k_1|R) + K_1(|k_1|R)}.$$

From the expression $G_k(R, R; \lambda) = -\frac{v_i^2(R)}{\lambda - \lambda_i} + G_1(R, R; \lambda)$, it follows that $G_k(R, R; \lambda)$ continuously varies from $-\infty$ to $+\infty$ when λ varies from λ_i to λ_{i+1}.

As value $\frac{K_1(|k_1|R)}{(|k_1|RK_0(|k_1|R) + K_1(|k_1|R))}$ is bounded, then there is at least one root of the equation $g(\lambda) = 0$ and this root lies between λ_i and λ_{i+1}.

Finally, it is necessary to prove that the term $v_i(R)$ does not vanish in expression $G_k(R, R; \lambda)$. Prove this from the contrary. Let $v_i(R) = 0$. Consider the Cauchy problem for the equation $\rho v_i'' + v_i' + \left(k_2^2(\rho)\rho - \rho^{-1}\right) v_i = \lambda_i \rho v_i$, $\rho \in [\delta, R]$, where $\delta > 0$, with initial conditions $v_i|_{\rho=R} = v_i'|_{\rho=R} = 0$. From the general theory of ordinary differential equations (see, for example, [10]) it is known that solution $v_i(\rho)$ of considered Cauchy problem exists and is unique for all $\rho \in [\delta, R]$. In this case, this solution coincides with function $v_i(R)$ for all $\rho \in [\delta, R]$. Function $v_i(R)$ is the function, which is contained in Green's function representation (10). On the other hand, the solution of the Cauchy problem for a linear equation with zero initial condition is the trivial solution. This contradicts to representation (10) of Green's function $G_k(\rho, \rho_0; \lambda)$ in the vicinity of $\lambda = \lambda_i$.

The following statement is the main result of this paper.

Statement 1. *Let $\varepsilon_1 \geqslant \varepsilon_0 > 0$, $\alpha > 0$ be real constants, and problem (11) have k eigenvalues λ_i such that the following inequalities $k_0^2\varepsilon_1 < \lambda_k < \lambda_{k-1} < \ldots < \lambda_2 < \lambda_1$ hold, where $k \geqslant 2$ be an* *integer. Then there is a value* $\alpha_0 > 0$ *such that for any* $\alpha \leqslant \alpha_0$ *the problem* P *has at least* $k-1$ *solutions* $\gamma_i \in (\sqrt{\lambda_{i+1}} + \delta_{i+1}, \sqrt{\lambda_i} - \delta_i)$, $i = \overline{1, k-1}$, *where* δ_i *are positive constants.*

Sketch of the proof. Let inequalities

$$k_0^2 \varepsilon_1 < \lambda_k < \lambda_{k-1} < \ldots < \lambda_2 < \lambda_1$$

hold, where $k \geqslant 2$, λ_i are solutions to (11).

We can choose sufficiently small $\delta_i > 0$ such that the Green function $G_k(\rho, \rho_0; \lambda)$ exists and is continuous on $\Gamma := \bigcup\limits_{i=1}^{k-1} \Gamma_i$, where $\Gamma_i := [\sqrt{\lambda_{i+1}} + \delta_{i+1}, \sqrt{\lambda_i} - \delta_i]$, $i = \overline{1, k-1}$ and the following inequality $g(\sqrt{\lambda_{i+1}} + \delta_{i+1}) \cdot g(\sqrt{\lambda_i} - \delta_i) < 0$ takes place.

It follows from the choice of δ_i that $F(\lambda)$ is bounded. Moreover, product $\alpha F(\lambda)$ can be made sufficiently small by choosing appropriate α. Consider the DE $\Phi(\lambda) = 0$. As it is shown before the function $g(\lambda)$ is continuous and reverses its sign when λ varies from $\lambda_{i+1} + \delta_{i+1}$ to $\lambda_i - \delta_i$. As the function $F(\lambda)$ is bounded then it is clear that the equation $\Phi(\lambda) = 0$ has at least $k-1$ roots $\tilde{\lambda}_i$, $i = \overline{1, k-1}$, if we choose appropriate α. Here $\tilde{\lambda}_i \in (\lambda_{i+1} + \delta_{i+1}, \lambda_i - \delta_i)$, $i = \overline{1, k-1}$.

From Statement 1 it follows that, under the above assumptions, there exist axially symmetrical propagating TE waves in cylindrical dielectric waveguides of circular cross-section filled with a nonmagnetic isotropic inhomogeneous medium with Kerr nonlinearity. This result generalizes the well-known similar statement for dielectric waveguides of circular cross-section filled with a linear medium ($\alpha = 0$).

It should be noticed that the value α_0 can be effectively estimated [1].

ACKNOWLEDGMENT

The work is partly supported by the Russian Federation President Grant (No. MK-90.2014.1), RFBR (No. 12-07-97010-r-A), and The Ministry of Education and Science of the Russian Federation (Goszadanie No. 2.1102.2014K).

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