

Algebraic Function Approximation for Eigenvalue Problem in Rectangular Waveguide Partially Filled with Transversely Magnetized Ferrite

Oguzhan Demiryurek¹ and Namik Yener²

¹Technical Education Faculty, Kocaeli University, Izmit, Kocaeli 41380, Turkey

²Technology Faculty, Kocaeli University, Izmit, Kocaeli 41380, Turkey

Abstract— In this work, a lossless and closed rectangular waveguide partially filled with a slab of transversely magnetized ferrite, without assuming vanishing of the derivative in the direction of the dc biasing field, has been analyzed using Method of Moment (MoM). Generally, there are basically three types of modes in the waveguide: propagating, evanescent and complex modes. To support evanescent mode, the waveguide must be bidirectional. But a rectangular waveguide filled with a slab of ferrite transversely magnetized is not bidirectional, if there exists no symmetry with respect to the rotation by π about an axis perpendicular to propagation direction. So such a waveguide does not support the evanescent waves. Propagation constant is complex or pure imaginary. Our waveguide is filled with a transversely magnetized gyrotropic medium and does not satisfy the condition of symmetry with respect to the rotation by π about an axis perpendicular to propagation direction. So it is not bidirectional and hence does not support evanescent modes. Results of the MoM are consistent with this situation. In a lossless and closed rectangular waveguide partially filled with a slab of transversely magnetized ferrite, Maxwell's equations, consisting of partial differential equations, are transformed into an infinite linear algebraic equation system by application of the Galerkin version of Moment method. As the result of algebraic operations, from the transmission line equations a quadratic eigenvalue problem is obtained whose eigenvalue corresponds to the propagation constant. The determinant of the coefficient matrix of the quadratic eigenvalue problem is a monic polynomial of the propagation constant whose coefficients are rational functions of the complex frequency. If the polynomial is set equal to zero and is multiplied by the common denominator of the coefficients, an algebraic equation is obtained. The roots of this equation are the propagation constants. In this work as the contribution, using algebraic function theory, the solution of the algebraic equation (propagation constant) for the waveguide filled with transversely magnetized ferrite medium, is expressed by means of a Puiseux series in the neighborhood of the algebraic branch point. Puiseux series coefficients are solved using transmission line equations and the propagation constant is computed from this series expansion. Puiseux series results are compared and found to conform with MoM results and the exact solution.

1. INTRODUCTION

Determination of waveguide propagation characteristics and modeling are very important issues for designing of microwave circuits. This is quite difficult for structures which have gyrotropic media such as magnetized ferrite as the filling medium. Analysis is usually performed by numerical techniques or using some simplifications. The exact dispersion relation for rectangular waveguide partially filled with a slab of ferrite transversely magnetized was obtained by assuming no dependence on the direction of the dc biasing field H_0 [1, 2]. The relevant structure was also examined by numerical methods such as the finite difference method [3], the spectral method [4]. Barziali and Gerosa have derived the dispersion relation with no restriction on the dependence along the direction of the dc magnetic field [5].

In this work, rectangular waveguide partially filled with a slab of transversely magnetized ferrite has been analyzed using the MoM. There is coupling between transverse and longitudinal field components, due to the bias direction of the ferrite slab hence transfer coefficient matrices are nonzero. It should be noted that the transfer coefficient matrices of the transmission line equations are zero in [6]. Therefore this study is distinguished from and more general than others [6–8] which also used the MoM.

The structure of a rectangular waveguide with transversely magnetized ferrite slab is shown in Figure 1. The permeability tensor for the ferrite slab magnetized transversely is given as follows [1],

$$[\mu] = \mu_0 \begin{bmatrix} \mu_r & 0 & -j\kappa \\ 0 & 1 & 0 \\ j\kappa & 0 & \mu_r \end{bmatrix}, \quad \mu_r = 1 + \frac{\omega_0 \omega_m}{\omega_0^2 - \omega^2} \text{ ve } \kappa = \frac{\omega \omega_m}{\omega_0^2 - \omega^2}. \quad (1)$$

Here, $\omega_0 = \gamma H_0$ Larmor resonance frequency, $\omega_m = \gamma(4\pi M_s)$, $\mu_0 = 4\pi \times 10^{-7}$ [H/m], $\gamma = 2.8$ [MHz/Oe] gyromagnetic ratio, H_0 [Oersted] dc magnetic biasing field, $4\pi M_s$ [Gauss] the saturation magnetization and ω operating frequency.

In Section 2, method used for examination of waveguide was taken up. In Section 3, Puiseux series results are compared with the MoM results and the exact solution.

2. METHOD

Fields of a closed, lossless, uniform waveguide filled with heterogeneous and/or anisotropic medium may be expanded as a Fourier series using eigenfunctions of waveguide filled with homogeneous and isotropic medium. Hence Maxwell's equations, consisting of partial differential equations, are transformed into an ordinary differential equation system which is called transmission line equations. In the inner product the test function used is set equal to basis function and this method is known as the Galerkin version of the MoM [9]. If the fields' dependence of the z direction is taken as $e^{-\gamma(p)z}$ and substituted in the ordinary differential equation system, a linear algebraic equation system is obtained [10]. This is as follows

$$-\gamma(p) \begin{bmatrix} v(p) \\ i(p) \end{bmatrix} = \begin{bmatrix} M(p) & -Z(p) \\ -Y(p) & N(p) \end{bmatrix} \begin{bmatrix} v(p) \\ i(p) \end{bmatrix}. \quad (2)$$

Here $p = \sigma + j\omega$ is the complex frequency, $Z(p)$, $Y(p)$, $M(p)$ and $N(p)$ are the series impedance, the shunt admittance, the voltage transfer coefficient and the current transfer coefficient matrices per unit length, respectively. These matrices are $2\mathbf{N} \times 2\mathbf{N}$ dimensional square matrices. Actually, they have infinite dimensions. But we use only finite truncations of these matrices in the approximation of the physical problem. $\mathbf{N}$ denotes the number of empty waveguide mode functions used in Fourier series expansion. $2\mathbf{N} \times 1$ dimensional column vectors $v(p)$ and $i(p)$ are the transmission line voltages and currents. $Z(p)$ and $Y(p)$ are also Foster matrices [11]. The transfer coefficient matrices of $M(p) \neq 0$ and $N(p) \neq 0$ are not zero.

In (2), the eigenvalues of the coefficient matrix correspond to the propagation constants and can be calculated numerically. However numerical values do not allow us to have complete information about the behavior of the propagation constant. Using algebraic function theory, a functional approach can be brought about for the behavior of the propagation constant [6].

Assume that $Z^{-1}(p)$ and $Y^{-1}(p)$ are exists. If Equation (2) is arranged as in (3), a quadratic eigenvalue problem is obtained. Here, $\gamma_n(p)$ is n th eigenvalue and $v_n(p)$ is the eigenvector corresponding to $\gamma_n(p)$. In (3), all of expressions are functions of the complex frequency p . Therefore, for the sake of simplicity, p dependence has not been shown in (3). Here, $\mathbf{I}$ denotes identity matrix.

$$[\gamma_n^2 \mathbf{I} + \gamma_n (M + ZNZ^{-1}) - ZY + ZNZ^{-1}M] v_n = 0 \quad (3)$$

In (3), we can write $Q(\gamma, p)v_n(p) = 0$. If $v_n(p) \neq 0$, $G(\gamma, p) = \det[Q(\gamma, p)]$ must be zero to ensure existence of a solution for (3). Determinant of $Q(\gamma, p)$ is a monic polynomial of $\gamma(p)$ whose coefficients are rational function of p . If this polynomial is set equal to zero and is multiplied by the common denominator of the coefficients

$$g(\gamma, p) = a_0(p)\gamma^{4\mathbf{N}}(p) + a_1(p)\gamma^{4\mathbf{N}-1}(p) + \dots + a_{4\mathbf{N}-1}(p)\gamma(p) + a_{4\mathbf{N}}(p) = 0 \quad (4)$$

i.e., an algebraic equation is obtained whose coefficients are entire rational functions of p . Roots of the algebraic equation correspond to the eigenvalues of (3) that is, propagation constants. These roots can be expressed using the series expansion about analytical and singular points. An algebraic equation can have only algebraic singularities [12]. These singularities can be poles or pole branch points which are zeros of $a_0(p)$ the coefficient of $\gamma^{4\mathbf{N}}(p)$ in (4) and branch points which are zeroes of the discriminant of (4).

In this work, solutions of propagation constant will be obtained about the algebraic branch point where the discriminant of $g(\gamma, p) = 0$ is equal to zero. Let $j\omega_B$ be an algebraic branch point on the $p = j\omega$ axis. At this point, $\gamma_n(j\omega)$ have two multiple roots and $\gamma_n(j\omega_B)$ is finite. n th eigenvalue $\gamma_n(j\omega)$ can be modeled by a Puiseux series expansion without negative power terms in the neighborhood of $j\omega_B$.

$$\gamma_n(j\omega) = \gamma_n(j\omega_B) + A_1 \sqrt{j\omega - j\omega_B} + A_2 (j\omega - j\omega_B) + \dots + A_m (j\omega - j\omega_B)^{m/2} + \dots \quad (5)$$

It is necessary to calculate the unknown coefficients of the series expansion, in order to determine the propagation constant numerically. A_1 in (5) was calculated through using various approximations for the derivatives of implicit function $G(\gamma, p)$.

2.1. Determination of A_1

An analytical expression will be obtained to determine A_1 in (5) using the results of expression of the eigenvalue equation in (3). Let $n = 1$ in (5). $\gamma_1(p)$ is second order multiple root at $p = j\omega_B$. Equation (5) can be arranged as follows.

$$\gamma_1(p) = \gamma_1(j\omega_B) + A_1 p' + A_2 (p')^2 + \dots + A_m (p')^m + \dots \quad (6)$$

Here $p' = \sqrt{p - j\omega_B}$. If Equation (6) is differentiated with respect to p' at $p' = 0$, the coefficient A_1 is obtained by,

$$A_1 = \left. \frac{d\gamma(p)}{dp'} \right|_{p'=0} = \left[-\frac{G_p}{G_\gamma} 2p' \right]_{\substack{p=j\omega_B \\ \gamma=\gamma_1(j\omega_B)}} \quad (7)$$

G_p and G_γ are derivatives of $G(\gamma, p)$ with respect to p and γ , respectively. G_γ will be vanish, when $p = j\omega_B$. In this case, we can express G_γ approximately in the neighborhood of $p = j\omega_B$.

$$G_\gamma = 2 [\gamma_1(p) - \gamma_1(j\omega_B)] E \quad (8)$$

Here $E = [\gamma_1(j\omega_B) - \gamma_3(j\omega_B)] [\gamma_1(j\omega_B) - \gamma_4(j\omega_B)] \dots [\gamma_1(j\omega_B) - \gamma_{4N}(j\omega_B)]$. From (6)

$$\gamma_1(p) - \gamma_1(j\omega_B) = A_1 p' + A_2 (p')^2 + \dots + A_m (p')^m + \dots \quad (9)$$

can be written. Using (7), (8) and (9) one has,

$$A_1 = \sqrt{-\frac{G_p}{E}} \quad (10)$$

G_p is equal to $G(\gamma, p)$ differentiated with respect to p it is found by employing standard formulae for the derivative of a determinant: $G_p = \text{trace}\{Adj[Q(\gamma, p)]\partial Q(\gamma, p)/\partial p\}$. Here, Adj denotes the adjugate matrix.

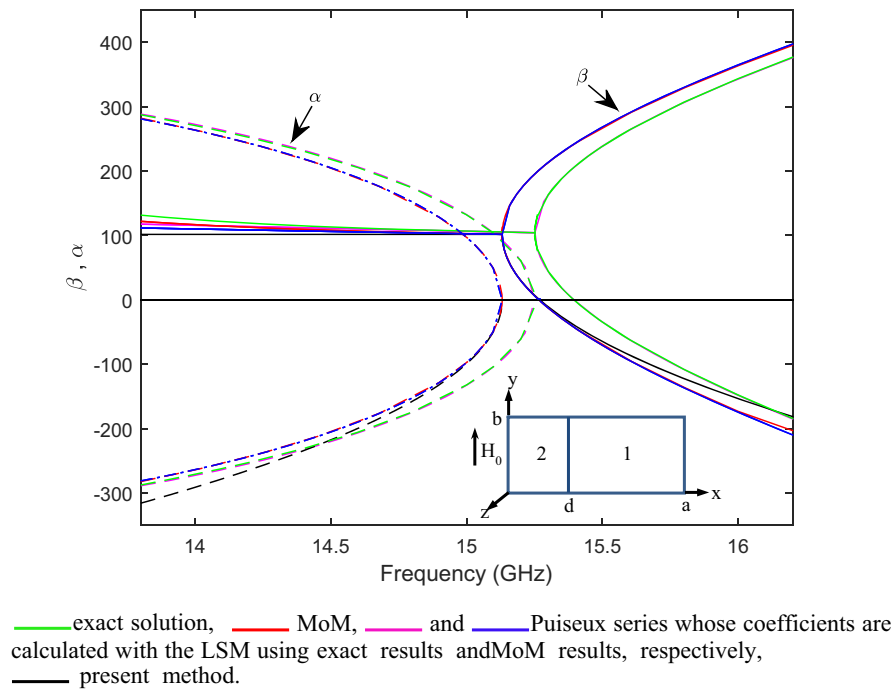


Figure 1: Phase coefficient (solid line) and attenuation constant (dashed line). Inset: cross-section of the rectangular waveguide partially filled with transversely magnetized ferrite. First region: (vacuum) ε_0, μ_0 , second region (ferrite): $\varepsilon_f, [\mu]$.

Table 1: Coefficients of Puiseux series computed by analytical and the least squares.

Coefficients	Analytical	LSM (used MoM results)	LSM (used exact results)
$\gamma(j\omega_B)$	$j101.66413$	$j101.66413$	$j104.2164$
A_1	$-2.4397 \times 10^{-3} (1 + j)$	$-2.41842 \times 10^{-3} (1 + j)$	$-2.39622 \times 10^{-3} (1 + j)$
A_2	-	-1.17966×10^{-9}	-1.462739×10^{-9}
A_3	-	$-2.9394 \times 10^{-14} (1 - j)$	$-2.86085 \times 10^{-14} (1 - j)$

3. NUMERICAL EXAMPLE

The parameter values of the rectangular waveguide partially filled with transversely magnetized ferrite are as follows: $4\pi M_s = 2000$ Gauss, $H_0 = 500$ Oersted, $a = 10$ mm, $b = 12.5$ mm, $d = 2$ mm, $\varepsilon_f/\varepsilon = 12.6$. In Figure 1, the results of Puiseux series method were compared with the results of the exact solution and the MoM in the neighborhood of the algebraic branch point.

Coefficient A_1 in the Puiseux series in (5), was calculated analytically as explained in Section 2.1. Puiseux series coefficients were also obtained using the least square method (LSM) in [13]. While this calculation was made, the results of the MoM and exact solution were used. The coefficients are shown in Table 1.

On the frequency axis backward wave is converted into evanescent mode in bidirectional waveguide. However, in Figure 1, there exists a transition from the backward wave to the forward wave at ω_C , i.e., evanescent mode is absent. To support the evanescent mode, waveguide must be bidirectional [14]. Our waveguide's structure does not support the symmetry condition in [14], therefore it is not bidirectional. The propagation constant computations using MoM also confirm this.

4. CONCLUSION

Eigenvalues were expressed by Puiseux series in the neighborhood of the algebraic branch point with the help of algebraic function theory. An analytical method is developed to calculate Puiseux series coefficients using the results of expression of the MoM for structures which can be represented by (2). These coefficients are also computed with the LSM using the results of exact solution and the MoM. The calculated coefficients are seen Table 1 to be consistent with each other. This confirms the validity of developed method.

As it seen in Figure 1, between the results of Puiseux series, exact solution and the MoM, a very close agreement has been achieved. This provides us with a capability to investigate the propagation problem about singular points in waveguide by algebraic function theory.

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