Diverging and Converging Beam Diffraction by a Wedge. Part II: Plane Wave Spectral Solutions and Complex Ray Solutions

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Abstract— In this present paper we present a novel solution for the diffraction of a converging beam. In that approach, the field of the incident converging beam is synthesized using the CS approach in conjunction with the plane wave spectral analysis, and the diffracted field is calculated using the appropriate integral representation expansion. The exact solution obtained via this spectral technique is then compared with an asymptotic solution obtained via CR tracing augmented with uniform complex ray diffraction. The CR solution includes also selection rules that delineate the CR lit and shadow zones for the diverging or converging beam cases. The overall goal of this research is the derivation of techniques for the analysis of 3D beam diffraction by a cone.

1. INTRODUCTION

The complex source (CS) method provides a canonical setting for the rigorous study of beam diffraction phenomena [1, 2]. In [3] we have used this approach for the analysis of two dimensional (2D) beam diffraction by a wedge as a function of the beam parameters: collimation, direction and displacement from the edge. However, as has been noted in [3], the straightforward CS formulation applies only for the case where the incident beam is diverging as it hits the edge, as schematized in Figure 1. In [4] we presented the novel solution that can be used for the case where the incident beam is converging (Figure 2). In this paper we extend this work by including the results of asymptotic complex ray tracing method augmented with uniform complex ray diffraction.

2. MODELING OF THE INCIDENT CONVERGING BEAM

We start with the conventional CS model in Figure 1 which, however, cannot be applied for the converging beam case of Figure 2. In this approach, the field is modeled as radiation from a point-source located at the complex coordinate point

$$\mathbf{r}_c = \mathbf{r}_0 + i\mathbf{b}, \quad \mathbf{r}_0 = (x_0, y_0) = (r_0, \phi_0), \quad \mathbf{b} = (x_b, y_b) = (b, \phi_b), \quad \mathbf{r}_c = (x_c, y_c) = (r_c, \phi_c) \tag{1}$$

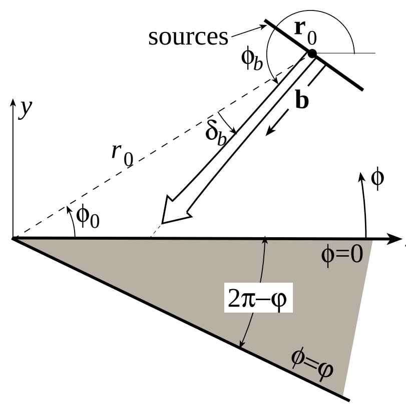

Figure 1: An incident diverging beam. A complex source beam (CSB; double arrow) emerges in the $\mathbf{b}$ direction from the physical source distribution centered at $\mathbf{r}_0$ whose length is $2b$ (thick line). The waist of the resulting beam is also located on that line and its size $\sqrt{b/k} \ll b$.

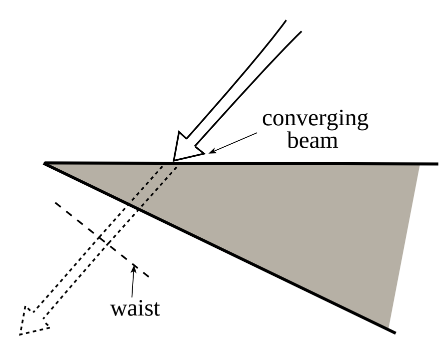

Figure 2: An incident converging beam. The beam propagates in the $\mathbf{b}$ direction, converging to a waist at $\mathbf{r}_0$, but it hits the wedge *before* it reaches that waist.

$\mathbf{r}_0$ and $\mathbf{b}$ are real vectors expressed in (1) in cartesian and polar forms. In free space, the field due to the source in (1) is obtained by an analytic continuation of the 2D free space Green's function

$$G(\mathbf{r},\mathbf{r}_c) = (i/4)H_0^{(1)}(ks), \qquad s = \sqrt{(x-x_c)^2+(y-y_c)^2}, \quad \operatorname{Re} s \geq 0, \tag{2}$$

$H_0^{(1)}(ks)$ is the Hankel function of the first kind and zero order and s is the complex distance from $\mathbf{r}_c$ to the real observation point $\mathbf{r}$.

As follows from the analysis above, the CS model in Figure 1 cannot describe the converging beam situation in Figure 2. In [4] this problem was circumvented by describing the incident converging beam as a plane wave integral, and then constructing the diffracted field due to these plane waves. The starting point is plane wave spectral representation of the solution in (2)

$$G(\mathbf{r},\mathbf{r}_c) = \frac{i}{4\pi}\int_{\mathcal{C}^+} d\xi\,\zeta^{-1}e^{ik[\xi\eta\pm\zeta(\sigma-ib)|]}, \quad \sigma \gtrless 0, \quad \zeta = \sqrt{1-\xi^2} \quad \text{with} \quad \operatorname{Im}\zeta \geq 0, \tag{3}$$

here (σ,η) are the beam coordinates centered at $\mathbf{r}_0$ such that σ is the coordinate *along* $\mathbf{b}$ while η is the *transversal* coordinate. The integration contour $\mathcal{C}^+$ extends along and passes above and below the branch points at $\xi = \mp 1$, respectively. The plane wave representation of the converging beam is obtained by using the upper sign in (3) for all σ, namely [4]

$$u^i(\mathbf{r};\,\mathbf{r}_c) = \frac{i}{4\pi}\int_{\mathcal{C}^\pm} \zeta^{-1}e^{ik\left[\xi\eta+\zeta(\sigma-ib)|\right]}d\xi, \quad \sigma \gtrless 0, \tag{4}$$

$\mathcal{C}^-$ is similar to $\mathcal{C}^+$ of (3) except that it passes below and above the branch points at $\xi = \mp 1$, respectively. $u^i(\mathbf{r};\,\mathbf{r}_c)$ in (3) describes a beam propagating along the σ-axis from $-\infty$ to ∞, having a waist at $\sigma = 0$ (i.e., at $\mathbf{r}_0$). It is convenient to put the integral in (4) in the form of an angular plane wave spectrum

$$u^i(\mathbf{r};\,\mathbf{r}_c) = \frac{i}{4\pi}\int_{\mathcal{P}^\pm} d\alpha\, e^{-ikr\cos(\alpha-\phi)}\tilde{u}^i(\alpha;\mathbf{r}_c), \quad \tilde{u}^i(\alpha;\,\mathbf{r}_c) = e^{ikr_c\cos(\alpha-\phi_c)}, \tag{5}$$

where the Sommerfeld integration pathes $\mathcal{P}^\pm$ extend from $-\pi/2\pm\infty$ to $\pi/2\mp\infty$, α is spectral angle measured from the σ- axis and $\tilde{u}^i$ is the plane-wave spectrum. ϕ_c is complex angle defined via $\phi_c = \arctan(y_c/x_c)$. We note now that if $\phi_0 \approx \phi_b - \pi$ then (5) describes an incident diverging beam, as in Figure 1, whereas if $\phi_0 \approx \phi_b$ then (5) describes a converging incident beam as in Figure 2.

3. CONVERGING BEAM DIFFRACTION BY A WEDGE

Referring to Figure 1 we consider an acoustically soft wedge with faces at $\phi = 0$ and $\phi = \varphi$ (Figure 1). Considering an incident plane wave $e^{-ikr\cos(\phi-\alpha)}$ where α is direction of the plane wave arrival, the total field can be expressed as [1]

$$u_{\mathrm{PW}}(\mathbf{r};\alpha) = \frac{-1}{8\pi}\int_{\mathcal{I}} dw\, e^{ikr\cos w}\; D(\phi,\alpha;w), \tag{6}$$

where $D(\phi,\alpha;w)$ is spectral diffraction coefficient. The total field due to the incident wave in (5) is given [1, 3]

$$u(\mathbf{r}) = \frac{i}{4\pi}\int_{\mathcal{P}^\pm} d\alpha\, u_{\mathrm{PW}}(\mathbf{r};\alpha)\tilde{u}^i(\alpha) = \frac{i}{4\pi}\int_{\mathcal{P}^\pm} d\alpha\tilde{u}^i(\alpha)\frac{-1}{8\pi}\int_{\mathcal{I}} dw e^{ikr\cos w}D(\phi,\alpha;w), \tag{7}$$

where $\tilde{u}^i(\alpha;\,\mathbf{r}_0,\phi_b)$ is the spectrum of the incident beam as defined in (5). The spectral diffraction coefficient $D(\phi,\alpha;w)$ introduces the lattice of poles with 2φ periodicity: $w_p = \phi\pm\alpha\mp\pi+2n\varphi$, with $n = 0,\pm1,\pm2,\ldots$. Assuming that the incident beam hits the wedge as is shown Figure 2, the most relevant poles are given by $w_p = \phi\pm\alpha-\pi$. Depends on the observation angle ϕ, these poles move across the integration path as angle $\phi\pm\alpha$ increases through π and give rise to the geometrical part of the solution of (7), coincides with the field of direct and reflected complex rays $u^{\mathrm{CR}}(\mathbf{r})$. Note that by extracting the contributions of these poles explicitly, the spectral integral in (6) may be regarded as the "diffracted field" $u^d_{\mathrm{PW}}(\mathbf{r})$ associated with the incident plane wave. Therefore, following [3] we introduce the "diffracted field" associated with the incident beam

$$u^d(\mathbf{r}) = \frac{i}{4\pi}\int_{\mathcal{P}^\pm} d\alpha\, u^d_{\mathrm{PW}}(\mathbf{r};\alpha)\tilde{u}^i(\alpha) = \frac{i}{4\pi}\int_{\mathcal{P}^\pm} d\alpha\tilde{u}^i(\alpha)\;\frac{-1}{8\pi}\int_{\mathcal{I}^{\mathrm{mod}}} dw e^{ikr\cos w}D(\phi,\alpha;w), \tag{8}$$

the integration path $\mathcal{I}^{\mathrm{mod}}$ extends along the imaginary w-axis [1, 3].

4. COMPLEX RAYS REPRESENTATION (CR)

In Reference [3], the complex ray representation of 2D beam diffraction by a wedge has been considered in the framework of the complex source (CS) approach. Here we extend this approach to the case of converging beam.

4.1. The Incident Field

The incident field of the converging beam can be treated by using complex geometrical optics. Thus the ingoing part of the incident converging beam is

$$u^i \sim i/4H_0^{(2)}(ks) \sim -\sqrt{\frac{1}{8\pi ks}}e^{-iks-i\pi/4}, \quad \mathrm{Re}\, s > 0, \tag{9}$$

where s is the complex *distance* and given by (2). The corresponding complex rays are

$$\mathbf{r} = \mathbf{r}_c - s\mathring{\mathbf{s}}, \tag{10}$$

where $\mathring{\mathbf{s}} = (\mathring{s}_x, \mathring{s}_y)$ is the complex ray direction such that $\mathring{s} \cdot \mathring{s} = 1$. These rays emanate from the observation point $\mathbf{r}$ and propagates toward the complex point $\mathbf{r}_c$. Beneath the waste $\mathbf{r}_0$ the ingoing beam field transforms into outgoing and is given by

$$u^i \sim i/4H_0^{(1)}(ks) \sim \sqrt{\frac{1}{8\pi ks}}e^{iks+i\pi/4}, \quad \mathrm{Re}\, s > 0, \tag{11}$$

and the corresponding rays are expressed by

$$\mathbf{r} = \mathbf{r}_c + s\mathring{\mathbf{s}}. \tag{12}$$

In this case the rays emanate from the complex point $\mathbf{r}_c$ and propagates toward the observation point $\mathbf{r}$.

4.2. The Field of the Direct Ray

To find the field scattered by a wedge, one needs the complex continuation of wedge geometry. This was done in [3] and we refer the reader to this work.

To find the field of the direct ray we calculate the distance $s(\mathbf{r})$ along the ray via (2) and then the ray direction $\mathring{\mathbf{s}}(\mathbf{r})$ via (10) for the real observation point lie above the waist or via (12) for the observation points lie beneath the waste. Using (10), (12) we find $\mathbf{q}_j$, the complex point where these rays intercept plane j defined above, and s_{q_j}, the distance from $\mathbf{r}_c$ to $\mathbf{q}_j$. Note that s_{q_j} is found uniquely via (10)–(12) and that it does not involve a square root.

Selection rule 1: This ray is included in the field representation if $\mathbf{q}_j$ does not belong to the complex extension of face j.

4.3. The Field of the Reflected Rays

In order to calculate the reflected complex rays it is convenient to consider the image points $\vec{\bar{\mathbf{r}}}_{cj}$ of $\mathbf{r}_c$ with respect to plane j. We introduce also the image point of the wais $\mathbf{r}_{0j}$. For the real observation points lie beneath the image waist $\mathbf{r}_{0j}$, the reflected rays have trajectories like to (10) $\mathbf{r} = \mathbf{r}_{cj} - s\mathring{\mathbf{s}}$. Next we calculate s with $\mathrm{Re}\, s > 0$ and $\mathring{s}$ in the same manner as described above for the incident ray. We also calculate the complex points $\bar{\mathbf{q}}_j$ where these rays intercept plane j, and the distances s_{q_j} from $\mathbf{r}_{cj}$ to $\mathbf{q}_j$.

Selection rule 2.1: $\bar{R}_j$ is included in the field representation only if $\mathbf{r}_{qj}$ belongs to the complex extension of face j and also $0 < \mathrm{Re}\{\bar{s}_{qj}\} < \mathrm{Re}\{\bar{s}_j\}$.

For the real observation points lie above the image waist $\mathbf{r}_{0j}$, the reflected rays have trajectories like to (12) $\mathbf{r} = \mathbf{r}_{cj} + s\mathring{\mathbf{s}}$, we calculate the corresponding s with $\mathrm{Re}\{s\} > 0$ and $\mathring{s}$ in the same manner as described in above. We also calculate the complex points $\bar{\mathbf{q}}_j$ where the continuation of them intercept plane j, and the distances s_{q_j} from $\mathbf{r}_{cj}$ to $\mathbf{q}_j$.

Selection rule 2.2: The ray is included in the field representation only if $\mathbf{r}_{qj}$ belongs to the complex extension of face j.

The reflected field ψ^r is given by (9)–(11) with appropriate s multiplied by the boundary reflection coefficient.

5. COMPLEX RAYS DIFFRACTION

The edge diffracted field due to incident beam is given by (8). Here we derive the uniform representation for the (8). We start with evaluation the internal (dw) integral. The integral has two critical points: the saddle point $w_s = 0$ and the poles $w_p = \phi \pm \alpha - \pi$. Since the poles cannot cross the integration path we can evaluate it by applying the saddle-point technique which yields

$$u^d(\mathbf{r}) = \frac{i}{16\pi^2}\sqrt{\frac{2\pi}{ikr}}e^{ikr}\int_{\mathcal{P}^\pm} d\alpha\, e^{ikr_c\cos(\alpha-\phi_c')}D(\alpha,\phi). \tag{13}$$

The last integral has two saddle points $\alpha_s = \phi_c, \pi - \phi_c$ and the poles $\alpha_p = \pi \pm \phi$. In the case of diverging beam the integration path $\mathcal{C}^+$ can be deformed into SDP path through the saddle point $\alpha_s = \phi_c$ and in the case of converging beam the integration path $\mathcal{C}^-$ can be deformed into SDP path trough the saddle point $\pi - \phi_c$. The process of the uniform evaluation is based on the method outlined in [3]. The spectral diffraction coefficient $D(\alpha, \phi)$ is resolved in two terms: pole term and regular term

$$D(\alpha,\phi) = \frac{\mathcal{P}_p}{\alpha-\alpha_p} + \mathcal{R}_p(\alpha;\phi), \tag{14}$$

where $\mathcal{P}_p \equiv \lim_{\,w\to\alpha_p}(\alpha-\alpha_p)D(\alpha,\phi)$ and $\mathcal{R}_p(\alpha,\phi) = D(\alpha,\phi) - \frac{\mathcal{P}_p}{\alpha-\alpha_p}$. The pole term can be integrated analytically and the regular term can be evaluated asymptotically via the saddle point technique.

The analysis yields for diverging beam

$$u^d(\mathbf{r}) = \frac{\mathcal{P}_p}{16\pi}\sqrt{\frac{2\pi}{ikr}}e^{ik\chi_p}\mathrm{erfc}(-is_p) - \frac{\mathcal{R}_p(\phi_c,\phi)}{8\pi k\sqrt{rr_c}}e^{ik(r+r_c)}, \tag{15}$$

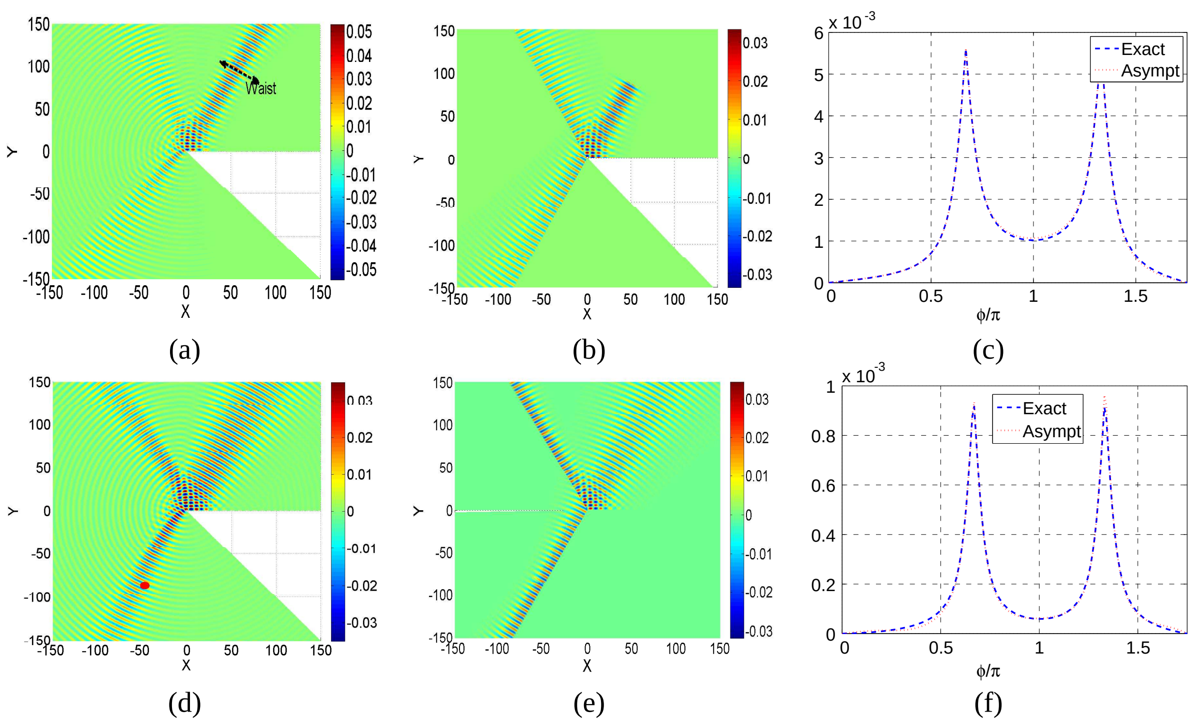

Figure 3: (a) The real part of the total field for the case of diverging beam calculated via (7). (b) The real part of the total field for the case of diverging beam calculated by CRT method. (c) The edge diffracted field for the case of diverging beam calculated at distance $r = 200$. The blue line represents the result obtained via the exact formulation (8) and the red line represents the results obtained via uniform asymptotic evaluation (15). Figures (d)–(f) likes to (a)–(c) but for the case of converging beam.

and for converging beam

$$u^d(\mathbf{r}) = \frac{\mathcal{P}_p}{16\pi}\sqrt{\frac{2\pi}{ikr}}e^{ik\chi_p}\operatorname{erfc}(-is_p) - \frac{\mathcal{R}_p(\pi-\phi_c,\phi)}{8\pi k\sqrt{rr_c}}e^{ik(r-r_c)}, \tag{16}$$

where χ_p is equals $\chi_p = -r_c\cos(\alpha_p - \phi_c)$ and s_p is given by $s_p = (\alpha_p - \phi_c)\sqrt{\frac{ikr_c}{2}}$ for diverging and $s_p = (\alpha_p - \pi + \phi_c)\sqrt{\frac{ikr_c}{2}}$ for converging beam.

6. NUMERICAL RESULTS

We consider diffraction by a 45° wedge (i.e., $\varphi = 1.75\pi$). The collimation length of the incident beam is $b = 80$ and its direction is $\phi_b = 240°$. The units are set such that $k = 1$. We consider the case of the diverging beam with the waist location $(r_0, \phi_0) = (100, 60°)$, and the case of converging beam with waist location $(r_0, \phi_0) = (100, 240°)$. In Figure 3 we depict the total field, calculated via (7), the total field calculated by asymptotic CRT method and the edge diffracted field for the case of diverging beam Figures 3(a), (b), (c) and for the case of converging beam Figures 3(d), (e), (f).

7. CONCLUSION

We derived a new solution for the diffraction of a converging beam by a wedge. In that approach, the field of the incident converging beam is synthesized using the CS technique in conjunction with the plane wave spectral analysis, and the diffracted field is then calculated using the spectral techniques. The final result in (7) may be applied for both cases where the incident beam is either *diverging* or *converging* as it hits the wedge, as schematized in Figures 1 and 2, respectively. The validity of the new solution has been verified by applying it first to the *diverging* beam case where the solution is valid [3], (see Figures 3(a), (b), (c)) before applying it to the *converging* beam case (Figures 3(d), (e), (f)). The long term goal of this research is the derivation of scattering and diffraction coefficients for a beam impinging on a tip of a cone.

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