Lorentz-like Transformations for the Velocity and Acceleration

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Abstract— Making use of the orthogonal transformation in the complex quaternion coordinate system, it is able to deduce directly invariants relevant to the radius vector and velocity, drawing out the Lorentz transformation and so forth, without the help of introducing the basic postulate. When the coordinate system transforms orthogonally from one to the other, the scalar part of and norm of one complex quaternion physical quantity will remain invariable respectively. Therefore the coordinate transformation between two three-dimensional coordinate systems, in the presence of relative movement, is equivalent to the orthogonal transformation between two complex quaternion coordinate systems. The above means that the paper is able to derive the Galilean and Lorentz transformations and so on for the radius vector, velocity, and acceleration. Especially it is not necessary to introduce additionally the invariant or basic postulate into the complex quaternion space, to deduce the coordinate transformations.

1. INTRODUCTION

In 1873 J. C. Maxwell was the first to apply the quaternion to describe the electromagnetic theory. This method spirits up subsequent scholars to adopt the complex quaternion to depict various aspects of electromagnetic fields, including Maxwell's equations, transmission of electromagnetic wave [1], force [2], and Lorentz transformation [3] and so on. In recent years some scholars applied the complex quaternion [4] to depict the Lorentz transformation and Special Theory of Relativity. M. M. Acevedo etc. figured out the Lorentz transformation [5] and so on by means of the quaternion. On the basis of the group of Lorentz transformation, I. Abonyi etc. constructed one pair of relativistic vectors and skew-symmetric tensors, representing a relativistic quaternion formulation of Maxwell's electrodynamics [6]. M. T. Teli investigated the quaternion representation of Lorentz transformation [7].

Analyzing the above researches reveals that all of these existing studies started off with some invariants (the space-time interval, or the speed of light), to deduce the Lorentz transformation and so forth. In other words, these existing studies possess the same distinguishing feature: they had to import additionally the invariant for deducing the coordinate transformation. However the paper is able to deduce some invariants, from the orthogonal transformation (or the rotational transform) of the complex quaternion coordinate system, inferring the Lorentz transformation and so forth. Obviously in the complex quaternion space, it is not necessary to introduce additionally the invariant to be the basic postulate for the Special Theory of Relativity.

In the physics, there is the relative movement between two three-dimensional coordinate systems, although the times in these two coordinate systems are different to each other. In the mathematics, this relative movement between two three-dimensional coordinate systems is equivalent to the orthogonal transformation between two quaternion coordinate systems. In the orthogonal transformation of quaternion coordinate system, the scalar remains invariable.

Making use of the above equivalency, the paper is capable of inferring invariants, related with the relative movement between two three-dimensional coordinate systems, including the scalar part of radius vector, norm of radius vector, scalar part of velocity, and norm of velocity. Further from the combination of these invariants, one can deduce the Galilean and Lorentz transformations and so on. That is, the paper can direct infer the invariant and Lorentz transformation, from the property of complex quaternion space. It is not necessary to introduce additionally the invariant, related with the coordinate transformation, into the complex quaternion space.

2. RADIUS VECTOR AND VELOCITY

In the complex quaternion space, there are two coordinate systems, $\mathbb{H}(ir_0, r_k)$ and $\mathbb{H}'(ir'_0, r'_k)$. In the complex quaternion coordinate system $\mathbb{H}$, the complex quaternion radius vector is, $\mathbb{R}(r_j) = ir_0\mathbf{i}_0 + \Sigma r_k\mathbf{i}_k$, the complex quaternion velocity is, $\mathbb{V}(v_j) = iv_0\mathbf{i}_0 + \Sigma v_k\mathbf{i}_k$. In the complex quaternion coordinate system $\mathbb{H}'$, the complex quaternion radius vector is, $\mathbb{R}'(r'_j) = ir'_0\mathbf{i}'_0 + \Sigma r'_k\mathbf{i}'_k$, the complex quaternion velocity is, $\mathbb{V}'(v'_j) = iv'_0\mathbf{i}'_0 + \Sigma v'_k\mathbf{i}'_k$. Herein r_j and r'_j are all real. i is the imaginary unit. $\mathbf{i}_0 = 1$, $\mathbf{i}_k^2 = -1$. $\mathbf{i}'_0 = 1$, ${\mathbf{i}'_k}^2 = -1$. $j = 0, 1, 2, 3$. $k = 1, 2, 3$.

Sometimes the complex quaternion coordinate system can be considered as the three-dimensional coordinate system with the time t. The paper will concentrate on the studying of one special case in the following context. There is the relative movement along one coordinate axis between two three-dimensional vector parts, $\mathbb{H}(r_k)$ and $\mathbb{H}'(r'_k)$, in the complex quaternion coordinate systems. According to the mathematical viewpoint, this relative movement is equivalent to the orthogonal transformation between two complex quaternion coordinate systems, $\mathbb{H}(ir_0, r_k)$ and $\mathbb{H}'(ir'_0, r'_k)$.

When the complex quaternion coordinate system orthogonally transforms from $\mathbb{H}(ir_0, r_k)$ to $\mathbb{H}'(ir'_0, r'_k)$, the scalar part of and norm of one complex quaternion physics quantity will remain invariant respectively. Making use of the scalar part of radius vector and of velocity, as well as the norm of radius vector and of velocity, it is able to deduce some invariants in the complex quaternion coordinate system for the radius vector and velocity. Further from a few different combinations of these invariants, the paper is capable of inferring some coordinate transformations.

2.1. Galilean Transformation

From the complex quaternion radius vector and velocity, it is capable of deducing one coordinate transformation, which is able to be degenerated into the Galilean transformation. In the complex quaternion coordinate transformation, according to the physical property, which the scalar part of radius vector and of velocity remain invariant respectively, there are,

$$r'_0 = r_0, \quad v'_0 = v_0, \tag{1}$$

where $r_0 = r_{00} + v_0 t$, $r'_0 = r'_{00} + v'_0 t'$. t and t' are respectively the time interval in $\mathbb{H}$ and $\mathbb{H}'$. r_{00} and r'_{00} are initial values. $v_1 = \partial r_1/\partial t$, $v'_1 = \partial r'_1/\partial t'$.

The above consists of two invariants in the complex quaternion space, and the coordinate transformation can be derived from these two invariants. When two coordinate systems possess one relative movement along the direction, $r_1\mathbf{i}_1$, with the relative speed, v_1, the coordinate transformation is,

$$r'_0 = r_0, \quad r'_1 = r_1 - r_0 v_1/v_0, \quad r'_2 = r_2, \quad r'_3 = r_3. \tag{2}$$

Choosing the initial values, $r_{00} = 0$ and $r'_{00} = 0$, the above is degenerated into the Galilean transformation in the Classical Mechanics as follows,

$$r'_0 = r_0, \quad r'_1 = r_1 - v_1 t, \quad r'_2 = r_2, \quad r'_3 = r_3. \tag{3}$$

2.2. Lorentz Transformation

From the norm of complex quaternion radius vector and the complex quaternion velocity, it is able to infer one coordinate transformation, which may be reduced into the Lorentz transformation. The norm of complex quaternion radius vector is,

$$S_R = \mathbb{R} \circ \mathbb{R}^* = -r_0^2 + \Sigma r_k^2, \tag{4}$$

where $*$ denotes the quaternion conjugate, and $\circ$ is the quaternion multiplication.

In a similar way, in the complex quaternion coordinate systems $\mathbb{H}'$, the norm of complex quaternion radius vector is, $S'_R = \mathbb{R}' \circ \mathbb{R}'^* = -{r'_0}^2 + \Sigma {r'_k}^2$.

In the complex quaternion coordinate transformation, according to the physical property, which the norm of radius vector and the scalar part of velocity remain invariant respectively, there are,

$$S'_R = S_R, \quad v'_0 = v_0. \tag{5}$$

The above covers two invariants in the complex quaternion space, and the coordinate transformation can be derived from these two invariants. The deduction of coordinate transformation is similar to that of the pseudo-orthogonal transformation in the Minkowski space.

When two coordinate systems possess one relative movement along the direction, $r_1\mathbf{i}_1$, with the relative speed, v_1, the coordinate transformation is,

$$r'_1 = (r_1 - r_0 v_1/v_0) \,/\, \left(1 - v_1^2/v_0^2\right)^{1/2}, \quad r'_2 = r_2, \tag{6}$$

$$r'_0 = (r_0 - r_1 v_1/v_0) \,/\, \left(1 - v_1^2/v_0^2\right)^{1/2}, \quad r'_3 = r_3. \tag{7}$$

Choosing the initial values, r_{00} and r'_{00}, to be all zero, the above is reduced into the Lorentz transformation in the Special Theory of Relativity,

$$r'_1 = (r_1 - v_1 t) \,/\, \left(1 - v_1^2/v_0^2\right)^{1/2}, \quad r'_2 = r_2, \tag{8}$$

$$t' = \left(t - r_1 v_1/v_0^2\right) \,/\, \left(1 - v_1^2/v_0^2\right)^{1/2}, \quad r'_3 = r_3. \tag{9}$$

The above is fit for the situation that the relative speed v_1 is comparative fast. When $v_0^2 \gg v_1^2$, the Lorentz transformation will be degenerated into the Galilean transformation.

Obviously in the mathematics, the Lorentz transformation can be considered as the combination of two parts of coordinate transformations. a) The first step. By means of the property of orthogonal transformation, the coordinate system transforms from $\mathbb{H}(ir_0, r_k)$ into $\mathbb{H}'(ir'_0, r'_k)$, and this part of coordinate transformation possesses the initial values. b) The second step. Making use of the property of parallel-movement transformation, two coordinate systems, $\mathbb{H}(ir_0, r_k)$ and $\mathbb{H}'(ir'_0, r'_k)$, are transformed into two new ones, $\mathbb{H}(it, r_k)$ and $\mathbb{H}'(it', r'_k)$, and this part of coordinate transformation possesses a part of initial values.

3. VELOCITY AND ACCELERATION

In the complex quaternion coordinate system $\mathbb{H}(iv_0, v_k)$, the complex quaternion acceleration is, $\mathbb{D}(d_j) = id_0\mathbf{i}_0 + \Sigma d_k \mathbf{i}_k$. In the complex quaternion coordinate system $\mathbb{H}'(iv'_0, v'_k)$, the complex quaternion acceleration is, $\mathbb{D}'(d'_j) = id'_0\mathbf{i}'_0 + \Sigma d'_k \mathbf{i}'_k$. Making use of the scalar part of velocity and of acceleration, and the norm of velocity and of acceleration, it is able to deduce the coordinate transformation and invariants, in the complex quaternion coordinate system for the velocity and acceleration. In other words, substituting the velocity and acceleration for the radius vector and velocity respectively in the above context, the paper is capable of inferring the Lorentz-like transformation and so forth for the velocity. Herein d_j and d'_j are all real.

3.1. Galilean-like Transformation

From the complex quaternion velocity and acceleration, it is able to deduce one coordinate transformation. In the complex quaternion coordinate transformation, according to the physical property, that the scalar parts of velocity and of acceleration remain invariable respectively, there are,

$$v'_0 = v_0, \quad d'_0 = d_0, \tag{10}$$

where $v_0 = v_{00} + d_0 t$, $v'_0 = v'_{00} + d'_0 t'$. v_{00} and v'_{00} are initial values. $d_1 = \partial v_1/\partial t$, $d'_1 = \partial v'_1/\partial t'$.

The above consists of two invariants in the complex quaternion space, and the coordinate transformation can be derived from these two invariants. This transformation is similar to the Galilean transformation. When two coordinate systems possess one relative movement along the direction, $v_1\mathbf{i}_1$, with the relative acceleration, d_1, the coordinate transformation is,

$$v'_0 = v_0, \quad v'_1 = v_1 - v_0 d_1/d_0, \quad v'_2 = v_2, \quad v'_3 = v_3. \tag{11}$$

Under the influence of the scalar part d_0 of acceleration, the scalar part v_0 of velocity may be varying. This is similar to the case of the scalar part r_0 of radius vector. That is, under the influence of the scalar part v_0 of velocity, the scalar part r_0 of radius vector may be varying. However for each time t, the scalar part v_0 of velocity remains invariant in the orthogonal transformation. Of course the scalar part v_0 of velocity may be altered for the different time t.

3.2. Lorentz-like Transformation

From the norm of complex quaternion velocity and the complex quaternion acceleration, it is able to infer one coordinate transformation. In the complex quaternion coordinate systems $\mathbb{H}$, the norm of complex quaternion velocity is, $S_V = \mathbb{V} \circ \mathbb{V}^* = -v_0^2 + \Sigma v_k^2$. And in the complex quaternion coordinate systems $\mathbb{H}'$, the norm of complex quaternion velocity is, $S'_V = \mathbb{V}' \circ \mathbb{V}'^* = -{v'_0}^2 + \Sigma {v'_k}^2$. In the complex quaternion coordinate transformation, according to the physical property, which the norm of velocity and the scalar part of acceleration remain invariable respectively, there are,

$$S'_V = S_V, \quad d'_0 = d_0. \tag{12}$$

The above covers two invariants in the complex quaternion space, and the coordinate transformation can be deduced from two invariants. This transformation is similar to the Lorentz transformation. When two coordinate systems possess one relative movement along the direction, $v_1\mathbf{i}_1$, with the relative acceleration, d_1, the coordinate transformation is,

$$v_1' = (v_1 - v_0 d_1/d_0) \,/\, \left(1 - d_1^2/d_0^2\right)^{1/2}, \quad v_2' = v_2, \tag{13}$$

$$v_0' = (v_0 - v_1 d_1/d_0) \,/\, \left(1 - d_1^2/d_0^2\right)^{1/2}, \quad v_3' = v_3. \tag{14}$$

The above is fit for the situation that the relative acceleration d_1 is comparative high. When $d_0^2 \gg d_1^2$, the above will be degenerated into the Galilean-like transformation.

In the complex quaternion space, one invariant (or basic postulate), applied by the Galilean and Lorentz transformations, is that the speed of light is invariable. And this basic postulate is applied in the Classical Mechanics and Special Theory of Relativity. However the above applies one different invariant (or basic postulate), that is, the speed of light is variable.

At one interface between two different optical media (such as, the gas and liquid), the speed of light is variable obviously. In the local bulky region of this interface, the transmitted light will vary gradually its speed from one value to the other. If the thickness of this interface is large enough, the variation of the speed of light should be in evidence. Modifying the thickness of interface between the gas and liquid, the observer may be able to measure, d_0, which is the derivative of the speed of light with respect to the time t.

Obviously the paper spreads the application range of coordinate transformations in the complex quaternion space. In the paper, the application range of coordinate transformations was extended from the Galilean and Lorentz transformations in the vacuum, into the Lorentz-like transformation in the waveguide.

4. RADIUS VECTOR, VELOCITY, AND ACCELERATION

In the complex quaternion coordinate system, making use of not only the scalar part of radius vector, velocity, and acceleration, but also the norm of radius vector, velocity, and acceleration, it is able to deduce some invariants in the complex quaternion coordinate system for radius vector and velocity. Further from a few different combinations of these invariants, the paper is capable of inferring one coordinate transformation as follows.

From the norm of complex quaternion radius vector and of complex quaternion velocity, and the complex quaternion acceleration, it is able to deduce one coordinate transformation. In the complex quaternion coordinate transformation, according to the physical property, the norm of radius vector, the norm of velocity, and the scalar part of acceleration remain invariable respectively, there are,

$$S_R' = S_R, \quad S_V' = S_V, \quad d_0' = d_0. \tag{15}$$

The above involves three invariants in the complex quaternion space, and the coordinate transformation can be derived from these three invariants. a) From the invariants, Eq. (12), it is capable of inferring the first part of coordinate transformation. When two coordinate systems possess one relative movement along the direction, $v_1\mathbf{i}_1$, with the relative acceleration, d_1 , and the coordinate transformations are, Eqs. (13) and (14); b) From the invariants,

$$S_R' = S_R, \tag{16}$$

it is able to draw out the second part of coordinate transformation. When two coordinate systems possess one relative movement along the direction, $r_1\mathbf{i}_1$, with the relative speed, v_1, the Lorentz transformation is,

$$r_1' = (r_1 - r_0 v_1/v_0) \,/\, \left(1 - v_1^2/v_0^2\right)^{1/2}, \quad r_2' = r_2, \tag{17}$$

$$r_0' = (r_0 - r_1 v_1/v_0) \,/\, \left(1 - v_1^2/v_0^2\right)^{1/2}, \quad r_3' = r_3, \tag{18}$$

where v_0 and v_0' are all constant, however $v_0' \neq v_0$.

Plugging the ratio (v_0'/v_0), from Eqs. (13) and (14), into the above is able to conclude the relation formulae between the times, T and T'. $T = r_0/v_0$, and $T' = r_0'/v_0'$. Obviously the time t and the velocity v_0 both are changeable. The above is fit for the situation, in which the relative speed v_1 and relative acceleration d_1 both are comparative high.

5. DISCUSSIONS

In the complex quaternion space, when the coordinate system transforms orthogonally, the scalar is invariable. This scalar includes the scalar part of and norm of one complex quaternion physics quantity.

According to the above property, it is able to deduce various coordinate transformations of the complex quaternion physics quantity. In the coordinate system for the radius vector and velocity, one infers the Galilean and Lorentz transformations for the complex quaternion radius vector. In the coordinate system for the velocity and acceleration, the paper concludes the Galilean-like and Lorentz-like transformations for the complex quaternion vector. Obviously in the coordinate system for the radius vector, velocity, and acceleration, the combinatorial coordinate transformation is comparative complicated.

It should be noted that the investigation for the coordinate transformations has examined only some simple cases in the complex quaternion space. Despite its preliminary characteristics, this study can clearly indicate that some invariants and the Lorentz transformation for the radius vector and so on can be derived from the orthogonal transformation of the complex quaternion system, revealing the influence of velocity on the radius vector. Meanwhile making use of the orthogonal transformation of complex quaternion system, the paper is capable of deducing the Galilean-like and Lorentz-like transformations and so on for the velocity and acceleration, uncovering the influence of acceleration on the velocity. In the following study, the research will concentrate on only the investigations and applications of the Lorentz-like transformation for the velocity and so on.

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REFERENCES

1. Edmonds, J. D., "Quaternion wave equations in curved space-time," *International Journal of Theoretical Physics*, Vol. 10, No. 2, 115–122, 1974.
2. Weng, Z.-H., "Angular momentum and torque described with the complex octonion," *AIP Advances*, Vol. 4, No. 8, 087103, 16 Pages, 2014.
3. Nakamura, T. K., "Lorentz transform of black body radiation temperature," *Europhysics Letters*, Vol. 88, No. 2, 20004, 4 Pages, 2009.
4. Grusky, S. M., K. V. Khmelnytskaya, and V. V. Kravchenko, "On a quaternionic Maxwell equation for the time-dependent electromagnetic field in a chiral medium," *Journal of Physics A: Mathematical and General*, Vol. 37, No. 16, 4641–4647, 2004.
5. Acevedo, M. M., J. López-Bonilla, and M. Sánchez-Meraz, "Quaternions, Maxwell equations and lorentz transformations," *Apeiron*, Vol. 12, No. 4, 371–384, 2005.
6. Abonyi, I., J. F. Bito, and J. K. Tar,, "A quaternion representation of the Lorentz group for classical physical applications," *Journal of Physics A: Mathematical and General*, Vol. 24, No. 14, 3245–3254, 1991.
7. Teli, M. T., "Quaternionic form of unified Lorentz transformations," *Physics Letters A*, Vol. 75, No. 6, 460–462, 1980.