Electromagnetic Forces in the Complex-octonion Curved Space

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Abstract— J. C. Maxwell applied the quaternion analysis to describe the electromagnetic theory. While A. Einstein depicted the gravitational theory by means of the tensor analysis and curved Four-spacetime. At present the scholars research the electromagnetic and gravitational features, making use of the complex-quaternion/octonion curved space. It is capable of defining the parallel translation and covariant derivative of the complex-octonion curved space, by means of the orthogonality of two complex octonions, depicting simultaneously the electromagnetic and gravitational features in the complex-octonion curved space. The study reveals that the connection coefficient and curvature of the complex-octonion curved space have an influence on the field strength and field source of the electromagnetic and gravitational fields, impacting the angular momentum, torque, and force, especially the electromagnetic force in the curved space.

1. INTRODUCTION

In order to define the component of one physics quantity in the curve coordinate system, it is necessary to introduce the tangent-frame component of one point in the curved space in the paper. Apparently the tangent frame belongs to the affine frame. In general the tangent-frame components are neither same length, nor perpendicular to each other. For the pseudo-Riemann space [1] in the General Theory of Relativity, the tangent-frame component may be the vector or the scalar. In the quaternion or octonion curved space, the tangent-frame component may be the quaternion or octonion respectively.

For the gravitational and electromagnetic theories in the curved space, not all affine frames are suitable to be chosen as the curve coordinate system. And it is essential to demarcate and filter out the appropriate affine frames farther. When we study the physics quantity and relevant features in the curved space, it is proper to stipulate to choose the orthogonal affine frame (or the orthogonal curve coordinate system), for the sake of reducing the correlative mathematical difficulty. Subsequently the physics quantity in the orthogonal affine frame can be transformed into that in one orthogonal and unitary affine frame, in order to draw a comparison between the physics quantity in the flat space with that in the tangent space of the curved space.

According to this arrangement, the gravitational and electromagnetic theories can be extended from the flat space into the curved space. Firstly, transform the gravitational and electromagnetic theories in the flat space into that under the orthogonal and unitary affine frame of the tangent space in the curved space; Secondly, transform the gravitational and electromagnetic theories in the orthogonal and unitary affine frame into that in the orthogonal affine frame; Thirdly, in the orthogonal affine frame, study the influence of the curved space on the force and so forth in the gravitational and electromagnetic fields.

2. COMPLEX-OCTONION CURVED SPACE

In the complex-quaternion flat space, the complex-quaternion radius vector is $\mathbb{R}_g(ih^0, h^1, h^2, h^3)$, with the basis vector being $\mathbf{i}_i$. Meanwhile, in the complex-S-quaternion flat space [2], the complex-S-quaternion radius vector is $\mathbb{R}_e(iH^0, H^1, H^2, H^3)$, with the basis vector being $\mathbf{I}_i$. In the complex-octonion flat space, two radius vectors, $\mathbb{R}_g(ih^0, h^1, h^2, h^3)$ and $\mathbb{R}_e(iH^0, H^1, H^2, H^3)$, can be combined together to become one complex-octonion radius vector,

$$\mathbb{R}(h^r) = \mathbb{R}_g + k_{eg}\mathbb{R}_e = ih^0\mathbf{i}_0 + h^p\mathbf{i}_p + ih^4\mathbf{i}_4 + h^{4+p}\mathbf{i}_{4+p}, \tag{1}$$

where $h^{j+4} = k_{eg}H^j$, and $\mathbf{i}_{j+4} = \mathbf{I}_j$. $\mathbf{i}_0^2 = 1$, and $\mathbf{i}_f^2 = -1$. $\mathbf{i}_r^2 = \mathbf{i}_r \circ \mathbf{i}_r$, for each subscript r. $\mathbb{R}_g = ih^0\mathbf{i}_0 + h^p\mathbf{i}_p$. $\mathbb{R}_e = iH^0\mathbf{I}_0 + H^p\mathbf{I}_p$. $\circ$ is the octonion multiplication. h^i and H^i are all real. $r, s, t, u = 0, 1, 2, 3, 4, 5, 6, 7$. $f = 1, 2, 3, 4, 5, 6, 7$.

According to the multiplication of octonion, the norm S is written as, $S^2 = \mathbb{R} \circ \mathbb{R}^*$. The differential, dS, of the norm is able to be chosen as the arc length of the complex-octonion (rather than the real-octonion) curved space. Obviously, in case the contribution of $\mathbb{R}_e$ can be neglected, this norm will accord with the requirement of space-time interval in the physics. Herein $*$ is the octonion conjugate.

In the complex-quaternion curved space for the gravitational field, the complex-quaternion radius vector is, $\mathbb{R}_g(iu^0, u^1, u^2, u^3)$, with the tangent-frame quaternion being $\mathbf{e}_i$. In the complex-S-quaternion curved space for the electromagnetic field, the complex-S-quaternion radius vector is, $\mathbb{R}_e(iU^0, U^1, U^2, U^3)$, with the tangent-frame S-quaternion being $\mathbf{E}_i$. In the tangent space of complex-octonion curved space, the complex-octonion radius vector is, $\mathbb{R}(u^r) = \mathbb{R}_g(iu^0, u^1, u^2, u^3) + k_{eg}\mathbb{R}_e(iU^0, U^1, U^2, U^3)$. Making use of the transforming, $u^{j+4} = k_{eg}U^j$ and $\mathbf{e}_{j+4} = \mathbf{E}_j$, the complex-octonion radius vector can be written as, $\mathbb{R}(u^r) = iu^0\mathbf{e}_0 + u^p\mathbf{e}_p + iu^4\mathbf{e}_4 + u^{4+p}\mathbf{e}_{4+p}$.

In the orthogonal and nonunitary affine frame, the metric of complex-octonion curved space is,

$$dR^2 = d\mathbb{R}^* \circ d\mathbb{R} = g_{rs}du^r du^s, \tag{2}$$

where the metric coefficient is, $g_{rs} = \mathbf{e}_r^* \circ \mathbf{e}_s$. The orthogonal tangent-frame octonion is, $\mathbf{e}_r = \partial\mathbb{R}/\partial u^r$. $\mathbf{e}_0$ is the scalar part, while $\mathbf{e}_f$ is the component of vector part. $\mathbf{e}_r^2$ is real and nonunitary. $\mathbf{e}_r^2 = \mathbf{e}_r \circ \mathbf{e}_r$, for each subscript r. $u^0 = v_0 t$, v_0 is the speed of light, and t is the time. u^i and U^i are all real.

However, in the complex-octonion curved space, what the paper will be involved in is, the mathematical manipulation between the quaternion operator (rather than the octonion operator) in the complex quaternion space, with the physics quantity of electromagnetic field in the S-complex quaternion space. So that the discussion of the space-time interval in the paper will be constrained to deal only with the component of the complex-quaternion radius vector, $\mathbb{R}_g$.

In one complex octonion space, when the product of two complex octonions, $\mathbb{G}(ig^0, g^f)$ and $\mathbb{Q}(iq^0, q^f)$, is equal to zero, that is, $\mathbb{G}^* \circ \mathbb{Q} = 0$, two octonions, $\mathbb{G}$ and $\mathbb{Q}$, are perpendicular to each other. This definition is called as the octonion orthogonality. Therefore it is able to define one parallel translation in the complex-octonion curved space, including the quaternion parallel translation in the complex-quaternion curved space. Herein g^r and q^s are all real.

In the complex-octonion curved space, the complex-octonion physics quantity $\mathbb{A}_1$, in the tangent space $\mathbb{T}_1$ of one point $\mathbb{M}_1$ on the complex-octonion manifold, can be disassembled in the tangent space $\mathbb{T}_2$ of the point $\mathbb{M}_2$ near $\mathbb{M}_1$. According to the definition of octonion orthogonality, $\mathbb{A}_1$ can be separated into the projection component $\mathbb{A}_2$ in $\mathbb{T}_2$, and the orthogonal component $\mathbb{G}_2$ perpendicular to $\mathbb{T}_2$. While $\mathbb{A}_2$ is named for the parallel translation from $\mathbb{A}_1$. And this definition is called as the octonion parallel translation. Especially, when the scalar parts of $\mathbb{A}_1$ and of $\mathbb{A}_2$ are all null, $\mathbb{A}_1$ and $\mathbb{A}_2$ both will be degenerated into the vectors. Further the orthogonality of octonion is reduced to that of vector, and the octonion parallel translation to the Levi-Civita parallel translation. When the complex octonion is reduced to the complex quaternion, the orthogonality of octonion is degenerated into that of quaternion, and the octonion parallel translation into the quaternion parallel translation (Table 1).

For the first-rank contravariant tensor $Y^s(\mathbb{Q})$ of one point $\mathbb{Q}$ in the complex-octonion curved space, the component of octonion covariant derivation with respect to the coordinate u^t is,

$$\nabla_t Y^s = \partial Y^s/\partial u^t + \Gamma^s_{rt}Y^r, \tag{3}$$

Table 1: Comparison of major characteristics in some flat and curved spaces.

Term	pseudo-Euclidean space	pseudo-Riemann space	Quaternion space	Octonion space
tangent space	vector space	vector space	quaternion space	octonion space
orthogonality	vector	vector	quaternion	octonion
parallel translation	normal	Levi-Civita	quaternion	octonion
tangent frame	vector/scalar	vector/scalar	quaternion	octonion
metric	scalar product among vectors and scalars	scalar product among vectors and scalars	scalar product of quaternions	scalar product of octonions
connection coefficient	Γ^i_{jk}	Γ^i_{jk}	Γ^i_{jk}	Γ^s_{rt}
covariant derivation	$\partial_k A^i$	$\nabla_k A^i$	$\nabla_k A^i$	$\nabla_t A^s$

where Γ^s_{rt} is the connection coefficient. $g^{us} = (g_{us})^{-1}$, and $\Gamma^s_{rt} = (1/2)g^{us}(\partial g_{ru}/\partial u^t + \partial g_{ut}/\partial u^r - \partial g_{tr}/\partial u^u)$. Y^s is real.

3. ELECTROMAGNETIC FIELD EQUATIONS

In the complex-octonion curved space, the electromagnetic potential had been transformed from the orthogonal and unitary rectangular coordinate system (flat space) to the orthogonal and nonunitary affine frame (tangent space). From the integrating function of field potential, $\mathbb{X}$, the octonion field potential, $\mathbb{A} = \mathbb{A}_g + k_{eg}\mathbb{A}_e$, is defined as,

$$\mathbb{A} = i\Box^\times \circ \mathbb{X}, \tag{4}$$

where $\mathbb{A} = i\Box^\times \odot \mathbb{X} + i\Box^\times \otimes \mathbb{X}$. $\mathbb{X} = \mathbb{X}_g + k_{eg}\mathbb{X}_e$. The gravitational potential, $\mathbb{A}_g(ia^0, a^1, a^2, a^3)$, is defined from the integrating function of gravitational potential, $\mathbb{X}_g$, that is, $\mathbb{A}_g = i\Box^\times \circ \mathbb{X}_g$. $\mathbb{A}_g = i\Box^\times \odot \mathbb{X}_g + i\Box^\times \otimes \mathbb{X}_g$. $i\Box^\times \odot \mathbb{X}_g$ and $i\Box^\times \otimes \mathbb{X}_g$ denote respectively the scalar and vector parts of $\mathbb{A}_g$. $\Box a^i = i\mathbf{e}_0\nabla_0 a^i + \delta^{pq}\mathbf{e}_p\nabla_q a^i$. $\nabla a^i = \delta^{pq}\mathbf{e}_p\nabla_q a^i$. $\mathbf{e}^i = g^{ij}\mathbf{e}_j$. $\mathbb{X}_g = x^j\mathbf{e}_j$. The gauge equation is chosen as, $\nabla \times (x^p\mathbf{e}_p) = 0$. $\mathbb{A}_g = ia + \mathbf{a}$. $a = a^0\mathbf{e}_0$, and $\mathbf{a} = a^p\mathbf{e}_p$. The integrating function, $\mathbb{X}_e$, of electromagnetic potential is one S-quaternion physics quantity, and $\mathbb{X}_e = X^j\mathbf{E}_j$. The electromagnetic potential is $\mathbb{A}_e(iA^0, A^1, A^2, A^3) = i\Box^\times \circ \mathbb{X}$. $\mathbb{A}_e = i\Box^\times \odot \mathbb{X}_e + i\Box^\times \otimes \mathbb{X}_e$. $i\Box^\times \odot \mathbb{X}_e$ and $i\Box^\times \otimes \mathbb{X}_e$ denote respectively the scalar and vector parts of $\mathbb{A}_e$. The gauge equation is chosen as, $\nabla \times (X^p\mathbf{E}_p) = 0$. $\mathbb{A}_e = i\mathbf{A}_Q + \mathbf{A}$. $\mathbf{A}_Q = A^0\mathbf{E}_0$, and $\mathbf{A} = A^p\mathbf{E}_p$. Apparently, in the affine frame (Table 2), the electromagnetic potential includes not only the physics quantity (in the complex-S-quaternion curved space) but also the spatial parameter of curved space (in the complex-quaternion curved space). $\times$ is the complex conjugate. a^i and A^i are all real.

The octonion field strength, $\mathbb{F} = \mathbb{F}_g + k_{eg}\mathbb{F}_e$, is defined as,

$$\mathbb{F} = \Box \circ \mathbb{A}, \tag{5}$$

where $\mathbb{F} = \Box \odot \mathbb{A} + \Box \otimes \mathbb{A}$. The gravitational strength, $\mathbb{F}_g(if^0, f^1, f^2, f^3)$, is defined as, $\mathbb{F}_g = \Box \circ \mathbb{A}_g$. $\mathbb{F}_g = \Box \odot \mathbb{A}_g + \Box \otimes \mathbb{A}_g$. The scalar part of $\mathbb{F}_g$ is, $\Box \odot \mathbb{A}_g = if^0\mathbf{e}_0$, and the vector part of $\mathbb{F}_g$ is, $\Box \otimes \mathbb{A}_g = f^p\mathbf{e}_p$. The gauge equation of gravitational potential is chosen as, $f^0 = 0$. The vector part of gravitational strength can be separated into two components, $f^p\mathbf{e}_p = i\mathbf{g}/v_0 + \mathbf{b}$. One component, $\mathbf{g}/v_0 = (\mathbf{e}_0\nabla_0) \circ \mathbf{a} + \nabla \circ a$, is relevant to the acceleration, while the other, $\mathbf{b} = \nabla \times \mathbf{a}$, is associated in the precession angular velocity [3]. The electromagnetic strength is, $\mathbb{F}_e(iF^0, F^1, F^2, F^3) = \Box \circ \mathbb{A}_e$. $\mathbb{F}_e = \Box \odot \mathbb{A}_e + \Box \otimes \mathbb{A}_e$. The scalar part of $\mathbb{F}_e$ is, $\Box \odot \mathbb{A} = iF^0\mathbf{E}_0$, and the vector part of $\mathbb{F}_e$ is, $\Box \otimes \mathbb{A}_e = F^p\mathbf{E}_p$. The gauge equation of electromagnetic potential is chosen as, $F^0 = 0$. The vector part of electromagnetic strength can be separated into two components, $F^p\mathbf{E}_p = i\mathbf{E}/v_0 + \mathbf{B}$. One component, $\mathbf{E}/v_0 = (\mathbf{e}_0\nabla_0) \circ \mathbf{A} + \nabla \circ \mathbf{A}_q$, is the electric field intensity, while the other, $\mathbf{B} = \nabla \times \mathbf{A}$, is the magnetic flux density. f^0 and F^0 are real, and f^p and F^p are the complex numbers.

The octonion field source, $\mu\mathbb{S} = \mu_g\mathbb{S}_g + k_{eg}\mu_e\mathbb{S}_e$, is defined as follows,

$$-\mu\mathbb{S} = -(\mu_g\mathbb{S}_g + k_{eg}\mu_e\mathbb{S}_e - i\mathbb{F}^* \circ \mathbb{F}/v_0) = (\Box + i\mathbb{F}/v_0)^* \circ \mathbb{F}, \tag{6}$$

where $-\mu_g\mathbb{S}_g = \Box^* \circ \mathbb{F}_g$, and $-\mu_e\mathbb{S}_e = \Box^* \circ \mathbb{F}_e$. $-\mu_g\mathbb{S}_g = \Box^* \odot \mathbb{F}_g + \Box^* \otimes \mathbb{F}_g$, while $-\mu_e\mathbb{S}_e = \Box^* \odot \mathbb{F}_e + \Box^* \otimes \mathbb{F}_e$. The scalar part of $\mathbb{S}_g$ is, $-\Box^* \odot \mathbb{F}_g/\mu_g = is^0\mathbf{e}_0$, which is associated in the mass density. And the vector part of $\mathbb{S}_g$ is, $-\Box^* \otimes \mathbb{F}_g/\mu_g = is^p\mathbf{e}_p$, which is relevant to the density

Table 2: The multiplication of the operator and the physics quantity in the complex-octonion curved space.

definition	expression meaning
$\nabla \cdot (\mathbf{e}_p a^p)$	$(\mathbf{e}_1 \cdot \mathbf{e}_1)\nabla_1 a^1 + (\mathbf{e}_2 \cdot \mathbf{e}_2)\nabla_2 a^2 + (\mathbf{e}_3 \cdot \mathbf{e}_3)\nabla_3 a^3$
$\nabla \times (\mathbf{e}_p a^p)$	$(\mathbf{e}_2 \times \mathbf{e}_3)(\nabla_2 a^3 - \nabla_3 a^2) + (\mathbf{e}_3 \times \mathbf{e}_1)(\nabla_3 a^1 - \nabla_1 a^3) + (\mathbf{e}_1 \times \mathbf{e}_2)(\nabla_1 a^2 - \nabla_2 a^1)$
$\nabla \circ (\mathbf{e}_0 a^0)$	$(\mathbf{e}_1 \circ \mathbf{e}_0)\nabla_1 a^0 + (\mathbf{e}_2 \circ \mathbf{e}_0)\nabla_2 a^0 + (\mathbf{e}_3 \circ \mathbf{e}_0)\nabla_3 a^0$
$(\mathbf{e}_0\nabla_0) \circ (\mathbf{e}_p a^p)$	$(\mathbf{e}_0 \circ \mathbf{e}_1)\nabla_0 a^1 + (\mathbf{e}_0 \circ \mathbf{e}_2)\nabla_0 a^2 + (\mathbf{e}_0 \circ \mathbf{e}_3)\nabla_0 a^3$
$\nabla \cdot (\mathbf{E}_p A^p)$	$(\mathbf{e}_1 \cdot \mathbf{E}_1)\nabla_1 A^1 + (\mathbf{e}_2 \cdot \mathbf{E}_2)\nabla_2 A^2 + (\mathbf{e}_3 \cdot \mathbf{E}_3)\nabla_3 A^3$
$\nabla \times (\mathbf{E}_p A^p)$	$(\mathbf{e}_2 \times \mathbf{E}_3)(\nabla_2 A^3 - \nabla_3 A^2) + (\mathbf{e}_3 \times \mathbf{E}_1)(\nabla_3 A^1 - \nabla_1 A^3) + (\mathbf{e}_1 \times \mathbf{E}_2)(\nabla_1 A^2 - \nabla_2 A^1)$
$\nabla \circ (\mathbf{E}_0 A^0)$	$(\mathbf{e}_1 \circ \mathbf{E}_0)\nabla_1 A^0 + (\mathbf{e}_2 \circ \mathbf{E}_0)\nabla_2 A^0 + (\mathbf{e}_3 \circ \mathbf{E}_0)\nabla_3 A^0$
$(\mathbf{e}_0\nabla_0) \circ (\mathbf{E}_p A^p)$	$(\mathbf{e}_0 \circ \mathbf{E}_1)\nabla_0 A^1 + (\mathbf{e}_0 \circ \mathbf{E}_2)\nabla_0 A^2 + (\mathbf{e}_0 \circ \mathbf{E}_3)\nabla_0 A^3$

of linear momentum. The electromagnetic source is $\mathbb{S}_e(iS^0, S^1, S^2, S^3)$. The scalar part of $\mathbb{S}_e$ is, $-\Box^* \odot \mathbb{F}_e/\mu_e = iS^0\mathbf{E}_0$, and is associated in the density of electric charge. And the vector part of $\mathbb{S}_e$ is, $-\Box^* \otimes \mathbb{F}_e/\mu_e = S^p\mathbf{E}_p$, and is relevant to the density of electric current. μ, μ_g, and μ_e are the coefficients.

4. OCTONION ANGULAR MOMENTUM

In the complex-octonion curved space, the octonion linear momentum, $\mathbb{P}(p^i, P^i) = \mathbb{P}_g + k_{eg}\mathbb{P}_e$, is defined from the octonion field source,

$$\mathbb{P} = \mu\mathbb{S}/\mu_g, \tag{7}$$

where the component of the octonion linear momentum, $\mathbb{P}$, in the complex quaternion space is, $\mathbb{P}_g = \{\mu_g\mathbb{S}_g - (i\mathbb{F}/v_0)^* \circ \mathbb{F}\}/\mu_g = p^0\mathbf{e}_0 + p^q\mathbf{e}_q$. And the component of the octonion linear momentum, $\mathbb{P}$, in the complex S-quaternion space is, $\mathbb{P}_e = \mu_e\mathbb{S}_e/\mu_g = P^0\mathbf{E}_0 + P^q\mathbf{E}_q$. p^i and P^i are all real.

Subsequently, in the complex-octonion curved space, the octonion angular momentum, $\mathbb{L}(l^i, L^i) = \mathbb{L}_g + k_{eg}\mathbb{L}_e$, can be defined from the octonion linear momentum and radius vector,

$$\mathbb{L} = (\mathbb{R} + k_{rx}\mathbb{X})^\times \circ \mathbb{P}, \tag{8}$$

where $\mathbb{L} = (\mathbb{R} + k_{rx}\mathbb{X})^\times \odot \mathbb{P} + (\mathbb{R} + k_{rx}\mathbb{X})^\times \otimes \mathbb{P}$. The component of the octonion angular momentum, $\mathbb{L}$, in the complex quaternion space is, $\mathbb{L}_g = (\mathbb{R}_g + k_{rx}\mathbb{X}_g)^\times \circ \mathbb{P}_g + k_{eg}^2(\mathbb{R}_e + k_{rx}\mathbb{X}_e)^\times \circ \mathbb{P}_e = l^0\mathbf{e}_0 + l^q\mathbf{e}_q$. And the component of the octonion angular momentum, $\mathbb{L}$, in the complex S-quaternion space is, $\mathbb{L}_e = (\mathbb{R}_g + k_{rx}\mathbb{X}_g)^\times \circ \mathbb{P}_e + (\mathbb{R}_e + k_{rx}\mathbb{X}_e)^\times \circ \mathbb{P}_g = L^0\mathbf{E}_0 + L^q\mathbf{E}_q$. l^i and L^i are all real.

5. OCTONION FORCE

In the complex-octonion curved space, the octonion torque, $\mathbb{W}(w^i, W^i) = \mathbb{W}_g + k_{eg}\mathbb{W}_e$, is defined from the octonion linear momentum,

$$\mathbb{W} = -v_0(\Box + i\mathbb{F}/v_0) \circ \mathbb{L}\ , \tag{9}$$

where $\mathbb{W} = v_0(\Box + i\mathbb{F}/v_0) \odot \mathbb{L} + v_0(\Box + i\mathbb{F}/v_0) \otimes \mathbb{L}$. The component of the octonion torque, $\mathbb{W}$, in the complex quaternion space is, $\mathbb{W}_g = -(i\mathbb{F}_g \circ \mathbb{L}_g + ik_{eg}^2\mathbb{F}_e \circ \mathbb{L}_e + v_0\Box \circ \mathbb{L}_g) = w^0\mathbf{e}_0 + w^q\mathbf{e}_q$. And the component of the octonion torque, $\mathbb{W}$, in the complex S-quaternion space is, $\mathbb{W}_e = -(i\mathbb{F}_g \circ \mathbb{L}_e + i\mathbb{F}_e \circ \mathbb{L}_g + v_0\Box \circ \mathbb{L}_e) = W^0\mathbf{E}_0 + W^q\mathbf{E}_q$. w^i and W^i are all real.

In the complex-octonion curved space, the octonion force, $\mathbb{N}(n^i, N^i) = \mathbb{N}_g + k_{eg}\mathbb{N}_e$, is defined from the octonion torque,

$$\mathbb{N} = -(\Box + i\mathbb{F}/v_0) \circ \mathbb{W}, \tag{10}$$

where $\mathbb{N} = (\Box + i\mathbb{F}/v_0) \odot \mathbb{W} + (\Box + i\mathbb{F}/v_0) \otimes \mathbb{W}$. The component of the octonion force, $\mathbb{N}$, in the complex quaternion space is, $\mathbb{N}_g = -(i\mathbb{F}_g \circ \mathbb{W}_g/v_0 + ik_{eg}^2\mathbb{F}_e \circ \mathbb{W}_e/v_0 + \Box \circ \mathbb{W}_g) = n^0\mathbf{e}_0 + n^q\mathbf{e}_q$. And the component of the octonion force, $\mathbb{N}$, in the complex S-quaternion space is, $\mathbb{N}_e = -(i\mathbb{F}_g \circ \mathbb{W}_e/v_0 + i\mathbb{F}_e \circ \mathbb{W}_g/v_0 + \Box \circ \mathbb{W}_e) = N^0\mathbf{E}_0 + N^q\mathbf{E}_q$. n^i and N^i are all real.

In the above, the term, $(\Box + i\mathbb{F}/v_0) \odot \mathbb{W} = n^0\mathbf{e}_0$, denotes the scalar part of the octonion force, $\mathbb{N}$. While the term, $(\Box + i\mathbb{F}/v_0) \otimes \mathbb{W} = n^q\mathbf{e}_q + k_{eg}(N^0\mathbf{E}_0 + N^q\mathbf{E}_q)$, indicates the vector part of the octonion force, $\mathbb{N}$. The real part of the term, $n^0\mathbf{e}_0$, is relevant to the mass continuity equation. While the real part of the term, $N^0\mathbf{E}_0$, is associated with the current continuity equation. For the term, $n^q\mathbf{e}_q$, its imaginary part is connected with the force, and corresponded with the linear acceleration. And that its real part is corresponded with the precession angular velocity.

In the complex-octonion curved space, the imaginary part of $(n^p\mathbf{e}_p)/2$ is the force density, in the gravitational and electromagnetic fields, that is,

$$\mathbf{f} = \mathrm{Im}\left\{(n^p\mathbf{e}_p)/2\right\}, \tag{11}$$

where the force density $\mathbf{f}$ includes the inertial force, gravitational force, electromagnetic force, energy gradient force, and additional force term caused by curved space. The additional force term is relevant to the connection coefficient and curvature and so forth of the curved space.

The electromagnetic force, $\mathbf{F}_e = -\mathbf{B} \times (S^q\mathbf{E}_q) + \mathbf{E} \circ (S^0\mathbf{E}_0)/v_0$, in the curved space is,

$$\mathbf{F}_e = -\left(B^2S^3 - B^3S^2\right)\mathbf{E}_2 \circ \mathbf{E}_3 - \left(B^3S^1 - B^1S^3\right)\mathbf{E}_3 \circ \mathbf{E}_1 \tag{12}$$

$$-\left(B^1S^2 - B^2S^1\right)\mathbf{E}_1 \circ \mathbf{E}_2 + \left(S^0E^p/v_0\right)\mathbf{E}_p \circ \mathbf{E}_0, \tag{13}$$

where the product, $\mathbf{E}_i \circ \mathbf{E}_j$, belongs to the complex-quaternion curved space, according to the multiplication of octonion.

When the octonion field potential is chosen as the first-rank tensor in the complex-octonion curved space, the octonion linear momentum, angular momentum, and torque will be involved in the some spatial parameters of the complex-octonion curved space (Table 3). It means that the curved space has an influence on the octonion force, including the force, the mass continuity equation, and the current continuity equation.

Table 3: Some definitions of the physics quantity relevant to the gravitational and electromagnetic fields in the complex-octonion curved space.

physics quantity	definition
radius vector	$\mathbb{R} = \mathbb{R}_g + k_{eg}\mathbb{R}_e$
integral function	$\mathbb{X} = \mathbb{X}_g + k_{eg}\mathbb{X}_e$
field potential	$\mathbb{A} = i\Box^\times \circ \mathbb{X}$
field strength	$\mathbb{F} = \Box \circ \mathbb{A}$
field source	$\mu\mathbb{S} = -(i\mathbb{F}/v_0 + \Box)^* \circ \mathbb{F}$
linear momentum	$\mathbb{P} = \mu\mathbb{S}/\mu_g$
angular momentum	$\mathbb{L} = (\mathbb{R} + k_{rx}\mathbb{X})^\times \circ \mathbb{P}$
octonion torque	$\mathbb{W} = -v_0(i\mathbb{F}/v_0 + \Box) \circ \mathbb{L}$
octonion force	$\mathbb{N} = -(i\mathbb{F}/v_0 + \Box) \circ \mathbb{W}$

6. CONCLUSIONS

In the complex-octonion curved space, from the definitions of octonion orthogonality, parallel translation, and covariant derivative, it is able to inference the octonion field potential, field strength, field source, linear momentum, angular momentum, torque, and force and so on in the gravitational and electromagnetic fields. The force consists of the inertial force, gravitational force, electromagnetic force, energy gradient, and additional force term caused by the complex-octonion curved space. The connection coefficient and curvature of the complex-octonion curved space may impact the additional force term. The study reveals that one may appraise the deviation amplitude of the complex-octonion curved space departure from its flat space, by means of the measurements of the field potential, field strength, and force and so forth.

It should be noted that the paper discussed only some simple cases about the influences of the complex-octonion curved space on the field potential, field strength, and force and so forth. However it clearly states that the connection coefficient, curvature and other spatial parameters of the curved space exert an influence on the physics features of gravitational and electromagnetic fields. In the following study, it is going to explore the impact of the force, in the strong gravitational and electromagnetic fields, on the movement statues of one charged objective in the complex-octonion curved space. Moreover, it may intend to describe the gravitational and electromagnetic theories with some generalized frames, and then transform the field theories into that in the orthogonal affine frame by means of the appropriate transformation.

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