A Short Note on the Optimization of Halbach Arrays Used as Magnetic Springs

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Abstract— Modern neodymium magnets are today widely used for a wide variety of purposes and some commercial types have an adhesive force of approximately 100 kg even though the dimensions are only $5 * 5 * 2.5$ cm. Therefore the idea of using permanent magnets as springs isn't uncommon. Here, the specific magnetization scheme often described as a "Halbach array" is analyzed in the use as elements of a magnetic spring. The Halbach array has (ideally) a one sided flux and here both analytical methods using Maxwell's stress tensor and numerical simulations using FEMM are applied to calculate the restorative force created for two such opposing magnetic structures. This vertical force is highly dependent upon the dimensions of the structures and the magnetization wavelengths in the materials. Thus, varying these parameters will greatly affect the characteristics of the magnetic spring (e.g., spring stiffness, maximum restorative force etc.). Different applications will require different physical dimensions and magnetization wavelengths in the material.

1. INTRODUCTION

Halbach arrays are magnetic structures with special magnetic schemes with rotating (as function of position along the array) direction of magnetization. They were first described in 1973 by J. C. Mallinson [1] but later rediscovered and further explained by K. Halbach in 1981 [2]. The particular magnetization leads to a structure having (ideally) a one sided flux (i.e., the magnetic field is concentrated to the enhancing side and the field strength is (ideally) zero on the opposing, cancellation side of the structure). This constitutes however not a magnetic monopole. Halbach arrays were originally envisioned to be used for more accurate reading and writing information on magnetic tapes (see [1]) but are today used in such diverse application as free-electron lasers (as "wigglers"), magnetic levitation of high-speed trains, magnetic bearings and in brushless motors.

Here, Halbach arrays are analyzed in the use of elements in a magnetic spring. A magnetic spring is most easily formed by having two magnets with the same poles facing each other (e.g., "south-south"). However, using two simple "bar magnets" is not an optimal usage of magnetic material as much of the fields, and thus potential energy, "produced" isn't involved in the formation of the force between the two structures, (i.e., the fields that are not formed in the gap between the two magnets are not involved in the creation of the force). However, this not true if Halbach arrays are used as they will have almost all of their produced magnetic fields in the gap between the two structures.

Halbach arrays, where originally described as a continuous change of the direction of magnetization [1] and later as a segmented array of individual small magnets [2]. However, it can be seen (e.g., [3]) that decreasing the width of the individual magnets in a discretized array and, thus, the difference in the direction of magnetization between two subsequent magnets (i.e., increasing the discretization of the array) will have the shape of the magnetic field formed to converge to that formed by a continuous array. (The difference in magnetization direction, between two subsequent magnets, that is required to approximate a discretized Halbach array with a continuous is roughly 20° [3].) Therefore we can, if the discretization is enough, use the calculations for the magnetic field from a continuous Halbach array to describe the force created between two discretized Halbach array. (For a discretized Halbach array, internal forces are created between the individual magnets, but as has been seen [3] these are not of great concern as the forces are not large enough that available adhesive glue types can't hold the individual magnets in place.)

As will be described below, if Halbach arrays are used in a magnetic spring then the behavior of the spring is greatly affected by physical dimensions and the magnetization wavelength of the Halbach arrays.

2. RESTORATIVE FORCE

Here, the analytical calculation of the restorative force created follows $\mathbf{M} \rightarrow \mathbf{B} \rightarrow \mathbb{T} \rightarrow \mathbf{F}$, i.e., the magnetizations in the material (for the two Halbach arrays) give the total field created in the gap between them. From this field Maxwell's stress tensor, on one of the two arrays, can be calculated and, thus, the force created between them.

We utilize two Halbach arrays, one above the other, both of dimensions α ("width" along $\hat{x}$), β ("height" along $\hat{y}$) and γ ("depth" along $\hat{z}$) and magnetization wavelength λ. Here, continuous arrays are used for the analyses but, as discussed above, these can approximated by discretized arrays. Assume, for now, that the upper array is situated (this will be shifted later) so the top surface is at $y = 0$ and bottom surface at $y = -\beta$. The upper Halbach array is given the magnetization

$$\mathbf{M} = M_0 \left(\hat{\mathbf{x}} \sin kx + \hat{\mathbf{y}} \cos kx\right) \tag{1}$$

Here, k is the wavenumber ($k = \frac{2\pi}{\lambda}$) and λ is the magnetization wavelength in the material. From $\mathbf{M}$ the magnetic flux density, $\mathbf{B}$, can be obtained via $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ and since $\nabla \cdot \mathbf{B} = 0$

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \tag{2}$$

In addition, as permanent magnets only have bound magnetic currents ($\mathbf{J}_{free} = 0$) $\nabla \times \mathbf{H} = 0$. We can therefore describe the magnetic field with the magnetic scalar potential alone $\mathbf{H} = -\nabla\phi$ [4] and we get

$$\nabla \cdot \mathbf{M} = \nabla \cdot \nabla\phi = \nabla^2\phi \tag{3}$$

This can be solved [5] using the magnetization and necessary boundary conditions for the structure. From this scalar potential the magnetic field around the Halbach array can be obtained, through $\mathbf{H} = -\nabla\phi$. The magnetic field on the enhancing side of the upper array (i.e., below it) then becomes

$$\mathbf{H}_1 = M_0 \left(1 - e^{k\beta}\right) e^{ky} \left[\hat{\mathbf{x}} \sin(kx) - \hat{\mathbf{y}} \cos(kx)\right] \tag{4}$$

where, M_0 is the maximum magnetization in the material which is related to the residual magnetism via $M_0 = \frac{B_r}{\mu_0\mu_r}$ [6]. In the same way, the lower Halbach array is given the magnetization (assume, as above, top surface at $y = 0$ and bottom surface at $y = -\beta$)

$$\mathbf{M} = M_0 \left(\hat{\mathbf{x}} \sin kx - \hat{\mathbf{y}} \cos kx\right) \tag{5}$$

which creates a enhanced magnetic field, above the lower array, of

$$\mathbf{H}_2 = -M_0 \left(1 - e^{-k\beta}\right) e^{-ky} \left[\hat{\mathbf{x}} \sin(kx) + \hat{\mathbf{y}} \cos(kx)\right] \tag{6}$$

Having the top surface of the lower Halbach array at $y = 0$ and shifting the upper array upward a distance h (and adjusting for the shift in coordinate system for the upper array, i.e., change y to $y - (h + \beta)$ in $\mathbf{H_1}$) the total magnetic field ($\mathbf{H_{tot}} = \mathbf{H_1} + \mathbf{H_2}$) in the gap becomes

$$\begin{aligned}\mathbf{H}_{tot} = {} & M_0 \sin(kx) \left[(1 - e^{k\beta})e^{k(y-(h+\beta))} - (1 - e^{-k\beta})e^{-ky}\right] \hat{\mathbf{x}} \\ & -M_0 \cos(kx) \left[(1 - e^{k\beta})e^{k(y-(h+\beta))} + (1 - e^{-k\beta})e^{-ky}\right] \hat{\mathbf{y}}\end{aligned} \tag{7}$$

To calculate the restorative force created when the two Halbach arrays are moved together we assume that the lower array is fixed and investigate the restorative force formed when the upper array is moved downward, i.e., h decreases. (As we have chosen opposing magnetizations, e.g., "South-South", this force will be directed upward on the upper array.) If we assume permanent magnets that move "slow" and don't induce electric fields (through $\nabla \times \mathbf{E} = -d\mathbf{B}/dt$) of any significance (see [3] for the justification of this) we form Maxwell's stress tensor (MST) using [4]

$$T_{ij} = \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right) \tag{8}$$

The force now becomes

$$\mathbf{F} = \oint_S \mathbb{T} \cdot d\mathbf{a} \tag{9}$$

For the situation investigated here, we are primarily interested in the vertical component of the restorative force (as any discplacement in the $\hat{x}$ or $\hat{z}$ direction can be handled by surrounding fixtures). We form the tensor for the lower, fixed, array and it was seen (see [7]) that this situation can, with good accuracy, be approximated with

$$F_y = \oint_S (\mathbb{T} \cdot d\mathbf{a})_y \approx \int_{top} T_{yy,\,top} da_{top} = \gamma \int_{x=0}^{\alpha} T_{yy,\,top} dx \tag{10}$$

Thus, the complicated surface integral can be reduced to a line integral on the top surface of this array. Using (8) the tensor becomes (in the gap)

$$T_{yy,\,top} = \frac{1}{\mu_0}\left(B_y B_y - \frac{1}{2}\left(B_x^2 + B_y^2\right)\right) = \frac{1}{2\mu_0}\left(B_y^2 - B_x^2\right) \tag{11}$$

Using (7) (along with $\mathbf{B} = \mu_0 \mathbf{H}$ and $M_0 = \frac{B_r}{\mu_0 \mu_r}$) $T_{yy,\,top}$ becomes at the top surface of the lower array (i.e., $y = 0$)

$$\left.\begin{aligned} T_{yy,\,top} &= \frac{B_r^2}{2\mu_0\mu_r^2}\left(\cos^2(kx)\theta - \sin^2(kx)\psi\right) \\ \theta &= \left(\left(1-e^{k\beta}\right)e^{k(-h-\beta))} + \left(1-e^{-k\beta}\right)\right)^2 \\ \psi &= \left(\left(1-e^{k\beta}\right)e^{k(-h-\beta))} - \left(1-e^{-k\beta}\right)\right)^2 \end{aligned}\right\} \tag{12}$$

The line integral (10) can then easily be calculated using *Wolfram Alpha* [8]

$$\begin{aligned} F_y &= \gamma\frac{B_r^2}{2\mu_0\mu_r^2}\int_{x=0}^{\alpha}\left(\cos^2(kx)\theta - \sin^2(kx)\psi\right)dx \\ &= \begin{bmatrix} c_1 = \theta - \psi \\ c_2 = \theta + \psi \end{bmatrix} = \gamma\frac{B_r^2}{2\mu_0\mu_r^2}\frac{1}{4k}\left(2k\alpha c_1 + c_2\sin(2k\alpha)\right) \\ &= \gamma\frac{B_r^2}{2\mu_0\mu_r^2}\left(\frac{\alpha c_1}{2} + \frac{\lambda}{8\pi}c_2\sin\left(\frac{4\pi\alpha}{\lambda}\right)\right) \end{aligned} \tag{13}$$

FEMM ("Finite Element Method Magnetics") is a numerical simulation software that has been shown to be accurate (see [3] or [9] for more information) and we can investigate the force formed in the situation and compare it with the analytical results (see Figure 1). As we can see, the analytical and simulated results results agree well.

3. OBSERVATIONS

As can be seen from Figure 1 the magnetization wavelength in the material greatly affects the behavior of the magnetic spring. For instance, decreasing the wavelength (compared to the widths, α, of the arrays, i.e., λ/α) will increase the maximum restorative force (very short gap distances) but will reduce this force for larger distances. Comparing to a normal mechanical spring for which the restorative force can be described with, e.g., $\mathbf{F} = -kx'\hat{\mathbf{x}}$ the behavior can be understood with having a spring stiffness, k, that isn't constant (although mechanical springs are only approximately constant). In addition, as can be seen in Figure 2, changing the height (i.e., β) of the two arrays also changes the characteristics of the force in a non-linear way that is dependent on the value of the factor λ/α. For instance, a magnetization wavelength of $\lambda/\alpha = 0.25$ will, quite quickly, increase the maximal restorative force when increasing β whereas larger λ/α will not. This is important as one can falsely be lead to believe that adding more magnetic material will linearly increase the restorative force (which is not true). Looking at the characteristics of the forces shown in the figures we can translate this to a spring stiffness that is dependent upon the magnetization wavelength, dimensions as well as the gap distance. Thus, this imagined spring stiffness is hardly linear, i.e., $k = k(\lambda/\alpha, \beta, h)$.

In addition, a normal mechanical spring that is disturbed from its relaxed state will (barring mechanical deformation) try to restore the spring to its natural, relaxed, state (i.e., form). For the magnetic spring investigated here this isn't so. If compressed, for the here given magnetizations, the restorative force will act to separate the two Halbach arrays but if pulled apart no restorative force acts on the two arrays to pull them back together again. (If the magnetization is changed as to have a situation of "North-south", then the two Halbach arrays will try to come together instead). The situation can be likened to having the natural, relaxed, state of an infinite gap distance.

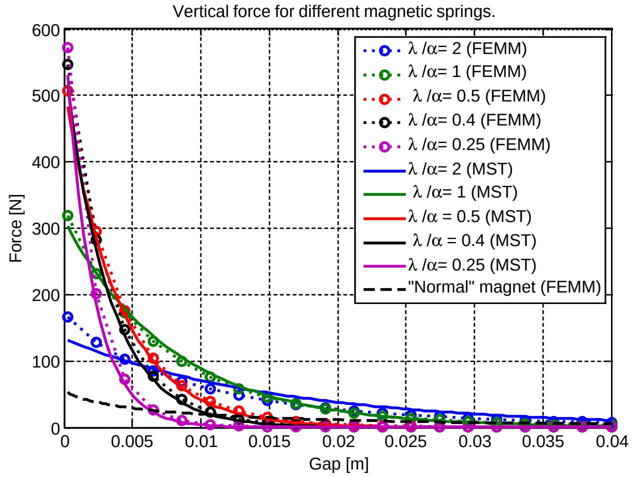

Figure 1: The vertical restorative force created (on the upper array) using two opposing Halbach arrays for cases of different wavelengths, λ (in relation to the width, α, of the Halbach arrays). The dimensions are here $\alpha = 5\,\mathrm{cm}$, $\beta = 1\,\mathrm{cm}$, $\gamma = 1\,\mathrm{cm}$ and material with B_r, of approximately 1.3 T and $\mu_r = 1.5$. The results are acquired both using numerical simulations (FEMM) and Maxwell's stress tensor (13). Both methods agree very well. Notice the much lower force (i.e., at most a factor of 10) created between two normal magnets of the same dimensions and maximum magnetization (M_0). The small differences, between the MST and FEMM results, in some cases, are due to the approximations made in order to acquire a simple analytical expression. (As explained in [7], the case where $\lambda/\alpha = 2$, the magnetic structures can't be considered to be a Halbach array anymore).

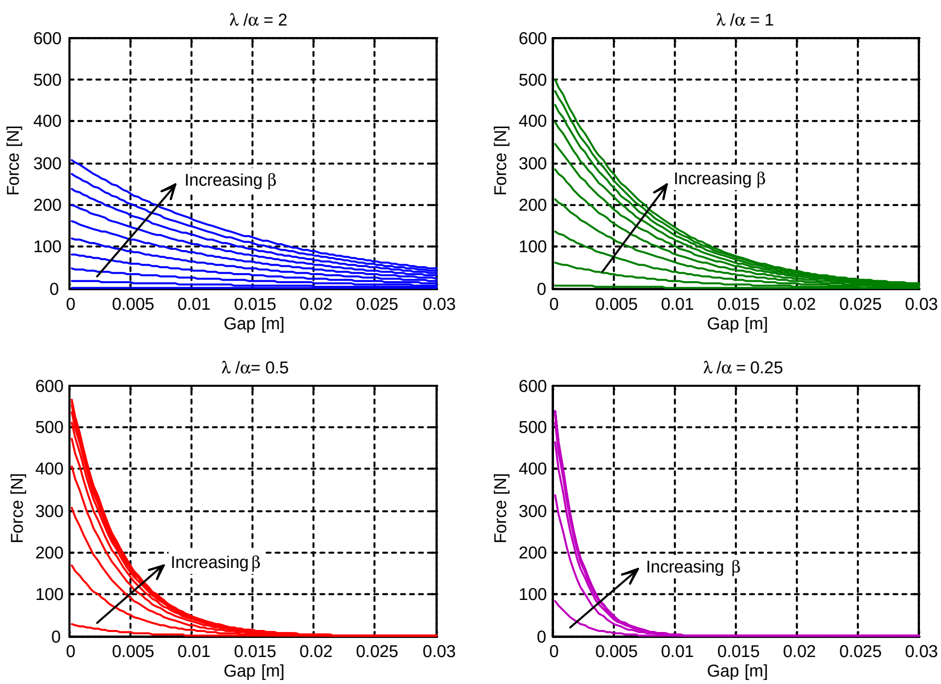

Figure 2: Using (13) the restorative force (on the upper array) is calculated for β having ten equally spaced values between $\beta = 0.1\,\mathrm{cm}$ and $\beta = 2\,\mathrm{cm}$. As can be seen, for different λ/α, the change in β affects the vertical force differently. The color scheme is the same as for the cases in Figure 1.

Furthermore, using two "normal" magnets of equal dimensions and maximum magnetization (M_0) will result in a much smaller restorative force that is, however, much more linear with the gap distance compared to using Halbach arrays. Thus, for different applications that require different characteristics the dimensions and magnetization wavelengths can be adapted to suit this.

4. CONCLUSION

Using two Halbach arrays with opposing magnetizations will form a magnetic spring. The characteristics of the restorative force for this magnetic spring will be highly non-linear and dependent on the magnetization wavelength and actual dimensions of the Halbach arrays. Different configurations (i.e., dimensions and magnetization wavelengths) can be made to fit different applications.

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