Inverse Problem Method for Permittivity Reconstruction of Two-layered Media: Numerical and Experimental Results

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Abstract— This study employs the technique developed in [1–3] and deals with complex permittivity reconstruction of layered materials in the form of diaphragms (sections) in a single-mode waveguide of rectangular cross section from the transmission coefficient measured at different frequencies.

1. INTRODUCTION

Reconstruction of electromagnetic parameters of nanocomposite and artificial materials is an urgent problem. In practice, as a rule, these parameters cannot be directly measured [4–6]. Therefore, methods of mathematical modeling must be applied. This work is a continuation of the series of papers [1–3] devoted to the analysis of permittivity determination of layered materials. We set forth a specially developed mathematical and numerical technique that combines analytical and numerical approaches providing thus the results with guaranteed accuracy.

We consider complex permittivity reconstruction of layered materials in the form of diaphragms (sections) in a single-mode waveguide of rectangular cross section from the transmission coefficient measured at different frequencies. The developed numerical-analytical method guarantees accuracy of the result and can be implemented in practical measurements.

2. INVERSE PROBLEM

Assume that a waveguide $P = \{x\colon\ 0 < x_1 < a, 0 < x_2 < b, -\infty < x_3 < \infty\}$ with the perfectly conducting boundary surface ∂P is given in the cartesian coordinate system. A three-dimensional body Q $(Q \subset P)$ $Q = \{x\colon\ 0 < x_1 < a, 0 < x_2 < b, 0 < x_3 < l\}$ is placed in the waveguide; the body has the form of a diaphragm adjacent to the waveguide walls.

Domain $P\backslash\bar{Q}$ is filled with an isotropic and homogeneous layered medium having constant permeability $(\mu_0 > 0)$ and constant permittivity $(\varepsilon_0 > 0)$, each section of the diaphragm having constant permeability μ_0 and permittivity:

$$\varepsilon_j(\omega) = \varepsilon_j^1 + i\frac{\sigma_j}{\omega} \tag{1}$$

where ε_j^1 is real part of complex permittivity and σ_j is conductivity $(j = 1, 2, \dots, n)$.

Length of each section l_j $(j = 1, 2, \dots, n)$ of the diaphragm is known.

The electromagnetic field inside and outside the object in the waveguide is governed by Maxwell's equations

$$\begin{aligned} \mathrm{rot}\mathbf{H} &= -i\omega\varepsilon\mathbf{E} \\ \mathrm{rot}\mathbf{E} &= i\omega\mu_0\mathbf{H}, \end{aligned} \tag{2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the vectors of the electric and magnetic field intensity and ω is the circular frequency.

Assume that $\pi/a < k_0 < \pi/b$, where $k_0^2 = \omega^2\varepsilon_0\mu_0$ (a single-mode waveguide). The incident electrical field is

$$\mathbf{E}^0 = \mathbf{e}_2 A \sin\left(\frac{\pi x_1}{a}\right) e^{-i\gamma_0 x_3} \tag{3}$$

with a known A and $\gamma_0 = \sqrt{k_0^2 - \pi^2/a^2}$.

Solving the forward problem for Maxwell's equations with the aid of (1) we obtain explicit expressions for the field inside every section of diaphragm Q and outside the diaphragm:

$$\begin{aligned} E^{(0)} &= \sin\left(\frac{\pi x_1}{a}\right)\left(Ae^{-i\gamma_0 x_3} + Be^{i\gamma_0 x_3}\right), \\ E^{(j)} &= \sin\left(\frac{\pi x_1}{a}\right)\left(C_j e^{-i\gamma_j x_3} + D_j e^{i\gamma_j x_3}\right), \\ E^{(n+1)} &= \sin\left(\frac{\pi x_1}{a}\right) Fe^{-i\gamma_{n+1} x_3}, \quad \gamma_{n+1} = \gamma_0 \\ & j = (1, 2, \dots, n) \end{aligned} \tag{4}$$

Substituting (4) into Maxwell's equations we have:

$$\gamma_j = \sqrt{\omega^2 \varepsilon_j^1 \mu_0 + \omega^2 \sigma_j \mu_0 - \frac{\pi^2}{a^2}} \tag{5}$$

From the conditions on the boundary surfaces $L: = \{x_3 = 0, \dots, x_j = l_j, \dots, x_3 = l_n\}$ of the diaphragm sections

$$[E]|_L = 0, \qquad [H]|_L = 0, \tag{6}$$

where square brackets $[\cdot]$ denote the function jump over the boundary surfaces, applied to (3)–(5) we obtain using conditions (6) a system of equations for the unknown coefficients

$$\begin{cases} A + B = C_1 + D_1 \\ \gamma_0 (B - A) = \gamma_1 (D_1 - C_1) \\ C_j e^{-i\gamma_j l_j} + D_j e^{i\gamma_j l_j} = C_{j+1} e^{-i\gamma_{j+1} l_j} + D_{j+1} e^{i\gamma_{j+1} l_j} \\ \gamma_j \left(D_j e^{i\gamma_j l_j} - C_j e^{-i\gamma_j l_j}\right) = \gamma_{j+1}\left(D_{j+1} e^{i\gamma_{j+1} l_j} - C_{j+1} e^{-i\gamma_{j+1} l_j}\right), \end{cases} \tag{7}$$

where $C_{n+1} = F,\ D_{n+1} = 0$. In system (7) coefficients A, B, C_j, D_j are supposed to be complex.

Inverse problem P: *find the complex permittivity $\varepsilon_j^1 + i\frac{\sigma_j}{\omega}$ of each section j from the known amplitude A of the incident wave and transmission coefficient F at different frequencies.*

Expressing C_j, D_j via C_{j+1}, D_{j+1} we obtain a recurrent formula that couples amplitudes A and F:

$$A = \frac{1}{2\prod\limits_{j=0}^{n} \gamma_j}(\gamma_n p_{n+1} + \gamma_0 q_{n+1}) F e^{-i\gamma_0 l_n}, \tag{8}$$

where

$$\begin{aligned} p_{j+1} &= \gamma_{j-1} p_j \cos\alpha_j + \gamma_j q_j i \sin\alpha_j, \quad p_1 := 1, \\ q_{j+1} &= \gamma_{j-1} p_j i \sin\alpha_j + \gamma_j q_j \cos\alpha_j, \quad q_1 := 1. \end{aligned} \tag{9}$$

Here $\alpha_j = \gamma_j(l_j - l_{j-1})$, $j = 1, \dots, n$.

We solve the inverse problem under study assuming that permittivities are governed by (1).

Considering the last equation at different frequencies we obtain a system of n equations with n unknown permittivities; solving this system numerically we find all permittivities.

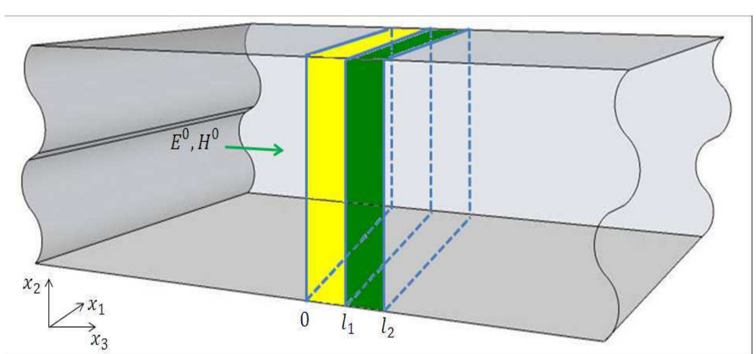

Figure 1: Two-sectional diaphragm in a waveguide.

3. TWO-SECTIONAL DIAPHRAGM

In the case of a two-sectional diaphragm, we obtain a system of 2 equations with 2 unknown permittivities by duplicating (7) at two different frequencies:

$$\begin{cases} \frac{A(\omega_1)}{F(\omega_1)} e^{i\gamma_0(\omega_1)l_2} = \frac{\gamma_2(\omega_1)p_3(\omega_1)}{2\gamma_0(\omega_1)\gamma_1(\omega_1)\gamma_2(\omega_1)} + \frac{\gamma_0(\omega_1)q_3(\omega_1)}{2\gamma_0(\omega_1)\gamma_1(\omega_1)\gamma_2(\omega_1)}, \\ \frac{A(\omega_2)}{F(\omega_2)} e^{i\gamma_0(\omega_2)l_2} = \frac{\gamma_2(\omega_2)p_3(\omega_2)}{2\gamma_0(\omega_2)\gamma_1(\omega_2)\gamma_2(\omega_2)} + \frac{\gamma_0(\omega_2)q_3(\omega_2)}{2\gamma_0(\omega_2)\gamma_1(\omega_2)\gamma_2(\omega_2)} \end{cases} \tag{10}$$

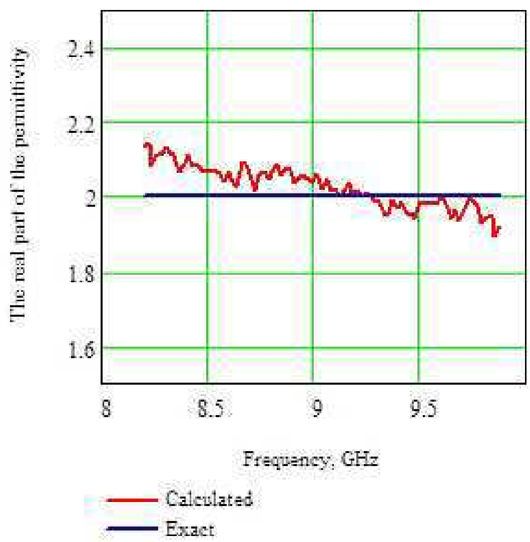

Figure 2: The real part of the permittivity of the first section ε_1^1. Blue and red lines denote, respectively, the permittivity values calculated by the present method and measured using experimental facilities.

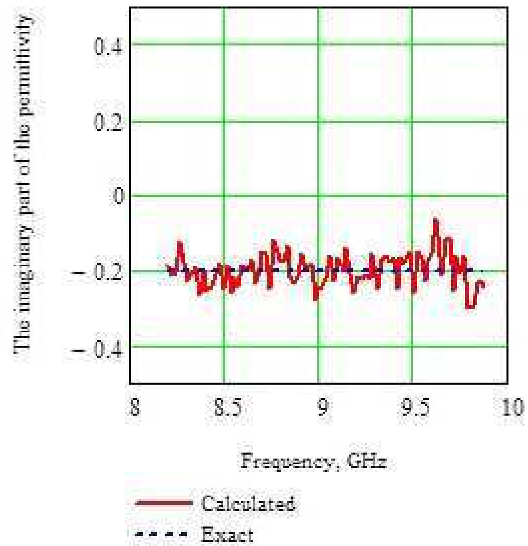

Figure 3: The conductivity of the first section σ_1. Blue and red lines denote, respectively, the permittivity values calculated by the present method and measured using experimental facilities.

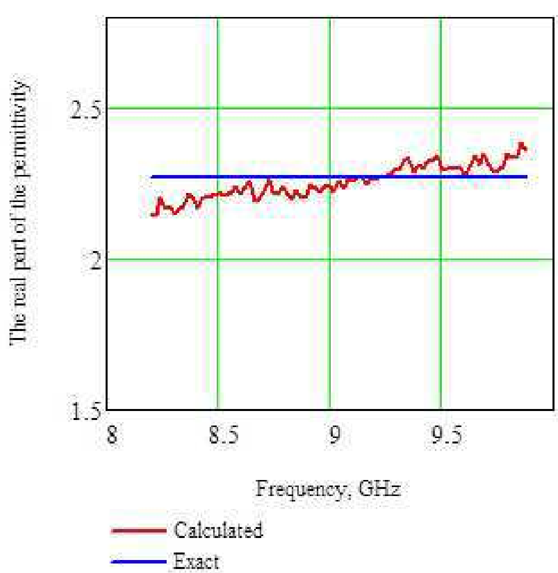

Figure 4: The real part of the permittivity of the first section ε_2^1. Blue and red lines denote, respectively, the permittivity values calculated by the present method and measured using experimental facilities.

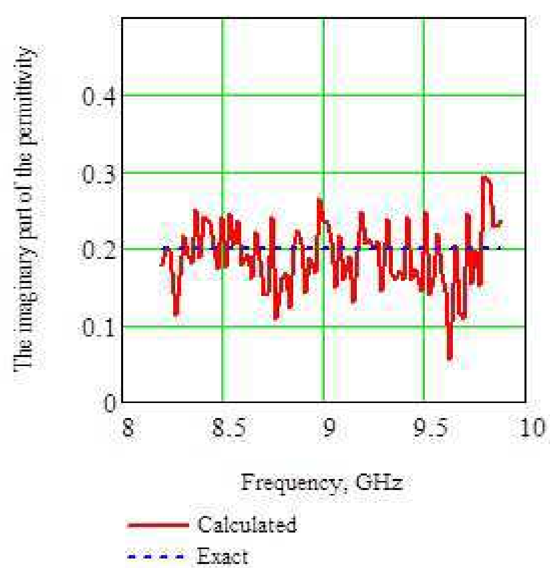

Figure 5: The conductivity of the second section σ_2. Blue and red lines denote, respectively, the conductivity values calculated by the present method and measured using experimental facilities.

where

$$\begin{aligned}
p_3 &= \gamma_1 p_2 \cos(\alpha_2) + \gamma_2 q_2 i \sin(\alpha_2), \\
p_2 &= \gamma_0 p_1 \cos(\alpha_1) + \gamma_1 q_1 i \sin(\alpha_1), \qquad p_1 = 1 \\
q_3 &= \gamma_1 p_2 i \sin(\alpha_2) + \gamma_2 q_2 \cos(\alpha_2), \\
q_2 &= \gamma_0 p_1 i \sin(\alpha_1) + \gamma_1 q_1 \cos(\alpha_1), \qquad q_1 = 1 \\
\gamma_j(\omega) &= \sqrt{\omega^2 \varepsilon_j^1 \mu_0 + \omega^2 \sigma_j \mu_0 - \frac{\pi^2}{a^2}}, \quad \alpha_j = \gamma_j(l_j - l_{j-1}), \quad j = 1, 2.
\end{aligned} \tag{11}$$

4. NUMERICAL RESULTS. TWO-SECTIONAL DIAPHRAGM

Parameters of the two-sectional diaphragm diaphragm are $a = 2.286\,\mathrm{cm}$, $b = 1.02\,\mathrm{cm}$, $c = 2\,\mathrm{cm}$, $l_1 = 0.995\,\mathrm{cm}$, $l_2 = 2.004\,\mathrm{cm}$, and the excitation frequency is $f \in (8, 2; 12, 4)$ GHz.

Numerical results confirm the efficiency of the method. In fact, the calculated permittivities coincide very well with the the statistical average (mean value) of the values obtained from the noisy measurement data, as it is illustrated by the Figures 1–5.

5. CONCLUSION

We have developed a numerical-analytical method of solution to the inverse problem of reconstructing permittivities of n-sectional diaphragms in a waveguide of rectangular cross-section. Comparison of numerical and experimental results validates the efficiency of the method.

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