Tailoring Specific Heat and Density in the Design of Thermal Transformation Media

Yu-Lin Tsai and Tungyang Chen
Department of Civil Engineering, National Cheng Kung University, Tainan 70101, Taiwan

Abstract— The transformation optics or acoustics in the design of metamaterials has been successfully applied to a number of physical phenomena. This novel method provides simple routes to attain certain significant effects such as invisibility cloaks, concentrators, and others. Given a mapping between two geometric configurations, the material properties of the transformed domain can be accordingly determined. They are in general anisotropic, position varying and sometimes of extreme values at certain points. This in fact restricts the practicality of real fabrication. Here we are concerned with the transient behavior of heat conduction. An inverse algorithm is presented here in the design of transformed medium based on the invariance of governing equation between physical and virtual configurations. We will demonstrate that the thermal capacity (thermal conductivity and heat capacity) can be tailored in specific forms, spatially uniform, linear or quadratic functions of positions, so that the design can be materialized more easily. Numerical illustrations are also presented for 2D and 3D thermal cloak for transient thermal conduction.

1. INTRODUCTION

In recent years, many significant achievements of invisibility cloaking have been motivated thanks to pioneering theoretical works [1, 2]. Recent advances in the field of metamaterials have extended beyond the manipulation of electromagnetic waves. A variety of cloaks working for different physical fields such as acoustic waves [3–6], and thermal dynamics [7]. The concept of the transformation method is based on the form invariance of the governing equation. Given a mapping between two configurations, the material properties of the transformed domain can be accordingly determined. However, most cloak realizations usually require extreme constitutive parameters (inhomogeneous, anisotropic, or even singular). In other words, this restricts the practicality of real fabrication. For feasible, several investigations have been demonstrated by using reduced cloak [8–10]. Another strategy for geometry transformations is quasi-conformal map [11–13]. It results in isotropic material parameters but requires a cloaking many times larger than the object to be hidden. Here we show a specific transformation can tailor the thermal capacity in specific form, spatially uniform, linear or quadratic functions of positions, so that the design can be materialized more easily.

A direct approach to analytically determined the material parameters of transformation shell ensures that the field exterior to shell will be the same as the original medium, as the governing equations remains unchanged and the continuous conditions are exactly fulfilled along the boundary [14]. In this paper, for transient thermal conductions, we propose a way to get the transformation function in an inverse manner when the thermal capacity is given, and then using the transformation function to get relative thermal conductivities.

2. TRANSFORMATION MEDIA FOR HEAT FLUX

The governing equation of heat flux without local source is given by $\nabla \cdot (\kappa \nabla T) = \rho C \partial T/\partial t$, where C is the specific heat, ρ is the material density and κ is the thermal conductivity. To demonstrate our algorithm, we use the cylindrical coordinates and write the components of κ as the relative conductivity with background media. For a time harmonic heat flux, the governing equation can be expressed as

$$\kappa_{rr}\frac{\partial^2 T}{\partial r^2}+\frac{\kappa_{\theta\theta}}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\kappa_{zz}\frac{\partial^2 T}{\partial z^2}+\frac{2\kappa_{r\theta}}{r}\frac{\partial^2 T}{\partial r\partial\theta}+\frac{2\kappa_{\theta z}}{r}\frac{\partial^2 T}{\partial\theta\partial z}+2\kappa_{rz}\frac{\partial^2 T}{\partial r\partial z}+\left(\frac{\kappa_{rr}}{r}+\frac{\partial\kappa_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\kappa_{r\theta}}{\partial\theta}+\frac{\partial\kappa_{rz}}{\partial z}\right)\frac{\partial T}{\partial r}$$
$$+\left(\frac{1}{r}\frac{\partial\kappa_{r\theta}}{\partial r}+\frac{1}{r^2}\frac{\partial\kappa_{\theta\theta}}{\partial\theta}+\frac{1}{r}\frac{\partial\kappa_{\theta z}}{\partial z}\right)\frac{\partial T}{\partial\theta}+\left(\frac{\kappa_{rz}}{r}+\frac{\partial\kappa_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\kappa_{\theta z}}{\partial\theta}+\frac{\partial\kappa_{zz}}{\partial z}\right)\frac{\partial T}{\partial z}=-\rho Ci\omega T. \quad (1)$$

Using the transformation media, an obstacle in the physical space can be formed as a different shape in the virtual space, in other words, transformation media makes an illusion. That is, to achieve any desired virtual reshaping effect provided an appropriate transformation can be found between the virtual space and the physical space. Here, we consider a coordinate transformation that maps the physical space $\mathbf{x}$ onto the virtual space $\mathbf{x}'$

$$r' = f\left(r, \theta, z\right), \quad \theta' = \theta, \quad z' = z. \tag{2}$$

Upon the transformation (2) and chain rule, the heat conduction Equation (1) can be transformed into virtual space as

$$\begin{aligned}
&\left[\kappa_{rr}\left(\frac{\partial f}{\partial r}\right)^2+\frac{\kappa_{\theta\theta}}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2+\kappa_{zz}\left(\frac{\partial f}{\partial z}\right)^2+\frac{2\kappa_{r\theta}}{r}\frac{\partial f}{\partial r}\frac{\partial f}{\partial \theta}+\frac{2\kappa_{\theta z}}{r}\frac{\partial f}{\partial \theta}\frac{\partial f}{\partial z}+2\kappa_{rz}\frac{\partial f}{\partial r}\frac{\partial f}{\partial z}\right]\frac{\partial^2 T}{\partial r'^2}\\
&+\frac{\kappa_{\theta\theta}}{r^2}\frac{\partial^2 T}{\partial \theta'^2}+\kappa_{zz}\frac{\partial^2 T}{\partial z'^2}+\left(\frac{2\kappa_{\theta\theta}}{r^2}\frac{\partial f}{\partial \theta}+\frac{2\kappa_{r\theta}}{r}\frac{\partial f}{\partial r}+\frac{2\kappa_{\theta z}}{r}\frac{\partial f}{\partial z}\right)\frac{\partial^2 T}{\partial r'\partial \theta'}+\frac{2\kappa_{\theta z}}{r}\frac{\partial^2 T}{\partial \theta'\partial z'}\\
&+\left(2\kappa_{zz}\frac{\partial f}{\partial z}+\frac{2\kappa_{\theta z}}{r}\frac{\partial f}{\partial \theta}+2\kappa_{rz}\frac{\partial f}{\partial r}\right)\frac{\partial^2 T}{\partial r'\partial z'}+\left[\kappa_{rr}\frac{\partial^2 f}{\partial r^2}+\frac{\kappa_{\theta\theta}}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\kappa_{zz}\frac{\partial^2 f}{\partial z^2}+\frac{2\kappa_{r\theta}}{r}\frac{\partial^2 f}{\partial r\partial \theta}+\frac{2\kappa_{\theta z}}{r}\frac{\partial^2 f}{\partial \theta\partial z}\right.\\
&+2\kappa_{rz}\frac{\partial^2 f}{\partial r\partial z}+\left(\frac{\kappa_{rr}}{r}+\frac{\partial \kappa_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \kappa_{r\theta}}{\partial \theta}+\frac{\partial \kappa_{rz}}{\partial z}\right)\frac{\partial f}{\partial r}+\left(\frac{1}{r}\frac{\partial \kappa_{r\theta}}{\partial r}+\frac{1}{r^2}\frac{\partial \kappa_{\theta\theta}}{\partial \theta}+\frac{1}{r}\frac{\partial \kappa_{\theta z}}{\partial z}\right)\frac{\partial f}{\partial \theta}\\
&\left.+\left(\frac{\kappa_{rz}}{r}+\frac{\partial \kappa_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \kappa_{\theta z}}{\partial \theta}+\frac{\partial \kappa_{zz}}{\partial z}\right)\frac{\partial f}{\partial z}\right]\frac{\partial T}{\partial r'}+\left(\frac{1}{r}\frac{\partial \kappa_{r\theta}}{\partial r}+\frac{1}{r^2}\frac{\partial \kappa_{\theta\theta}}{\partial \theta}+\frac{1}{r}\frac{\partial \kappa_{\theta z}}{\partial z}\right)\frac{\partial T}{\partial \theta'}\\
&+\left(\frac{\kappa_{rz}}{r}+\frac{\partial \kappa_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \kappa_{\theta z}}{\partial \theta}+\frac{\partial \kappa_{zz}}{\partial z}\right)\frac{\partial T}{\partial z'}+i\rho C\omega T=0.
\end{aligned} \tag{3}$$

To let the heat flux in the transformation device be effectively equivalent to the background material in virtual space. The governing equation of heat flux in background material can be written as

$$\frac{\partial^2 T'}{\partial r'^2}+\frac{1}{r'^2}\frac{\partial^2 T'}{\partial \theta'^2}+\frac{\partial^2 T'}{\partial z'^2}+\frac{1}{r'}\frac{\partial T'}{\partial r'}=-i\omega T'. \tag{4}$$

When the transformation media is effectively equivalent to that of the same domain filling with the background homogeneous material, it means that $T(r', \theta', z')$ equals to $T'(r', \theta', z')$ and then (3) should identical with (4). According to equate the coefficients in each term, we obtain the thermal capacity which are expressed by transformation function as

$$\begin{aligned}
\kappa_{rr} &= A^*\left(\frac{f+\frac{(\partial f/\partial \theta)^2}{f}+f\left(\frac{\partial f}{\partial z}\right)^2}{r\partial f/\partial r}\right), \quad \kappa_{r\theta}=-\frac{A^*}{f}\frac{\partial f}{\partial \theta}, \quad \kappa_{rz}=-\frac{A^* f}{r}\frac{\partial f}{\partial z},\\
\kappa_{\theta\theta} &= \frac{A^* r}{f}\frac{\partial f}{\partial r}, \quad \kappa_{\theta z}=0, \quad \kappa_{zz}=\rho C=\frac{A^* f}{r}\frac{\partial f}{\partial r},
\end{aligned} \tag{5}$$

where A^* is a constant which is determined by the boundary conditions. According to the continuous conditions, we demand the temperature and normal heat flux along the outer boundary must be the same for the two configurations (physical space Ω and virtual space Ω'), that are $T|_{\partial\Omega} = T|_{\partial\Omega'}$ and $\kappa\partial T/\partial n|_{\partial\Omega} = \kappa'\partial T'/\partial n'|_{\partial\Omega'}$. Therefore, the undetermined coefficient A^* is obtained as 1. In this paper, we consider the geometry of the transformation media as a cylinder, so the transformation relation in (2) would be simplified as $r' = f(r)$, $\theta' = \theta$, $z' = z$. And then, the material parameters of the transformation media are

$$\kappa_{rr}=\frac{f}{r\partial f/\partial r}, \quad \kappa_{\theta\theta}=\frac{r\partial f/\partial r}{f}, \quad \kappa_{zz}=\frac{f}{r}\frac{\partial f}{\partial r}, \quad \kappa_{r\theta}=\kappa_{rz}=\kappa_{\theta z}=0 \text{ and } \rho C=\frac{f\partial f/\partial r}{r}. \tag{6}$$

3. TAILOR A SPECIFIC THERMAL CAPACITY

From (6), we can tailor the thermal capacity in specific forms, uniform, linear or quadratic functions of positions, etc., by appropriate transformation function. Let thermal capacity be an arbitrary function of positions, $G(r)$, then we suppose the capacity as

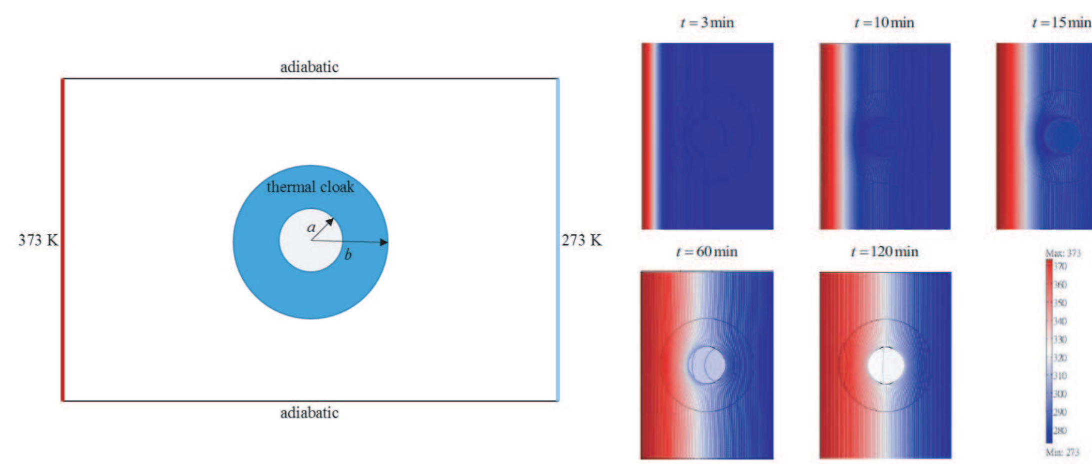
$$\rho C=\frac{f\partial f/\partial r}{r}=G\left(r\right). \tag{7}$$

First, we consider a uniform thermal capacity. To proceed, we suppose the capacity as α. By integration, the transformation function between to configurations is formed as $f(r) = \sqrt{\alpha r^2 + \xi}$. Since the outer boundary remains unchanged, $f(a) = 0$, and inner boundary shrinks into a point in the virtual space, $f(b) = b$, where a and b are the radii of the inner and outer boundary, then the coefficients α and ξ are obtained as $\alpha = b^2/\left(b^2 - a^2\right)$ and $\xi = -a^2 b^2/\left(b^2 - a^2\right)$. Therefore, the thermal conductivities of the transformation media in (6) become

$$\kappa_{rr} = \frac{r^2 - a^2}{r^2}, \quad \kappa_{\theta\theta} = \frac{r^2}{r^2 - a^2}, \quad \kappa_{zz} = \frac{b^2}{b^2 - a^2}, \quad \kappa_{r\theta} = \kappa_{rz} = \kappa_{\theta z} = 0 \text{ and } \rho C = \frac{b^2}{b^2 - a^2}. \tag{8}$$

Corresponding the finite element simulations, uniform hot and cold side temperature boundary conditions of 373 K and 273 K, respectively, were applied to the structure shown in Fig. 1(a). To demonstrate the validity of the cloak design, we consider the inner radius as $a = 0.2$ m and the outer radius as $b = 0.5$ m. Fig. 1(b) depicts the snap-shot of our cloak at various times t. In our simulation, the conductivity of background media is 400 W/m-K. The contours of temperature display curved line in cylindrical cloak medium. After passing through the cloak, the potential fields outside of the cloaking region remain undisturbed and the fluxes return to their original trajectories without changing direction and magnitude. However, in the long-time limit or static, as shown in Fig. 1(b) ($t = 120$ min), the to-be-protected inner region does eventually heat up.

The thermal capacity also can be a linear function of positions, for example, we alter the arbitrary function $G(r)$ of (7) as $G(r) = \zeta r$. Similar as the processing of uniform capacity, the transformation function is obtained as $f = (b^2(r^3 - a^3)/(b^3 - a^3))^{1/2}$ by the geometry boundary conditions. At last,



(a) (b)

Figure 1: (a) Illustration of the 2D cylindrical cloaking numerical simulation. The to-be-protected region ($r < a$) is coated by a cloaking device. (b) Snapshots of the transient thermal conductions for each time. The contours of temperature are bent in the cloaking region.

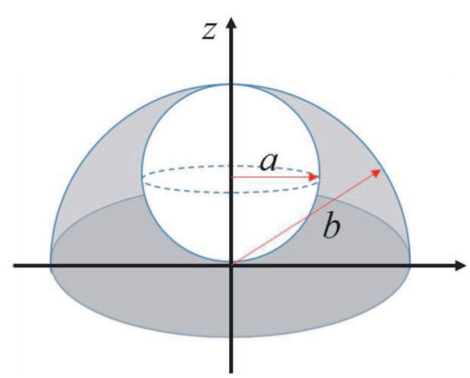


Figure 2: A schematic of the 3D cloaking device. There is a ball-region with radius a embedded in the half-sphere with radius b.

we design a 3D thermal cloaking device with a uniform thermal capacity for each layer. Fig. 2 illustrates that a sphere with radius a is embedded in a half-sphere with radius b. According to the outer boundary remains unchanged and the inner boundary shrinks into zero, the coefficient α and ξ are determined as $(b^2-z^2)/(b^2-2az)$ and $(b^2-a^2)(z^2-2az)/(b^2-2az)$, respectively. Then the transformation for this 3D thermal cloak is obtained as $f(r,z)=\sqrt{(b^2-z^2)(r^2+z^2-2az)/(b^2-2az)}$.

4. CONCLUSION

In this paper, we propose a way to obtain a specific heat conductivity and capacity for a transient thermal flux. In manufacture, although some investigations have proposed multilayer structures to approach the anisotropic and inhomogeneous material parameters, the fabrication is still difficult because the capacity tends to infinite at inner boundary. When the capacity is a constant, the fabrication becomes much easier. We also demonstrate a 3D cloaking device with a specific capacity varied by z direction. For fabrication, it can be built by 3D printing technology.

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