A Comparative Study of Analytical and Numerical Analysis for Coaxial Probe Aperture in a Dissipative Media

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Abstract— This paper presents an analysis of the coaxial probe aperture immersed in a dissipative medium using Levine's integral calculation and Nevels's MoM routines. The analytical integration and MoM are programmed with MATLAB code. The comparison between simplicity and accuracy of both methods has been discussed in detail. This paper intends to enhance the compatibility study of analytical and numerical methods for coaxial probe, since literature on this study for coaxial probe is less compared to linear monopole.

1. INTRODUCTION

Analytical and numerical modeling is a tool that cannot be lacked in the current communication educational, research and engineering world. Communication devices can be designed and modified in a short time based on the modeling calculated results without building an actual prototype and repeatedly testing its performance by using measurement method. Directly, the cost and time in the design of communication devices, such as antenna, filter and directional coupler can be saved. However, the main purpose of this study is for new researchers involved in the communication to better understand the development background and concepts of numerical modeling other than use only. In additional, a good understanding of the mathematical principles of wave propagation and radiation is an essential requirement for undergraduate in communication engineering, especially for student who prepares to further their postgraduate study. Although, relevant information in this regard can be found in many textbooks, mostly are on the modeling of a linear monopole (or dipole), very little information on the open-ended coaxial probe. Hence, in this paper, a brief review of analytical and method of moment (MoM) based numerical modeling are particularly focused on a coaxial probe.

There are close relationship between the analytical and numerical modeling, due to the fact that MoM cannot be used if problems have no established analytical integral equation as a governing function. Moreover, the accuracy of the MoM method is dependent on the rigors of the integral equation. Fortunately, there are several rigorous quasi-static integrals have been derived, such as Levine's [1] and Nevels's [4] admittance integral formulations. The MoM, a technique that is suitable to be used to solve the unbounded scattering wave problems, and requires no special boundary treatment, since those integral equations are expressed in terms of Green's functions, which are naturally satisfied the radiated fields. In fact, the difference between the analytical and numerical modeling are the complexity of the solutions. In engineering field, the analytical solutions require various assumptions in the theoretical equation. For instance, the difference in the voltages between the coaxial gap (at $a < \rho < b$) at aperture surface (at $z = 0$) is assumed to be 1 V (or 1/2 V) for analytical solution as shown in Figure 1(a). However, in numerical solution, the difference voltage is an unknown parameter. For instance, the MoM routines may divide the aperture surface into small element areas and the unknown difference voltages at each area are solved one by one [3, 4] as shown in Figure 1(b). Thus, the distribution of voltage difference at the aperture surface can be virtualized obviously. The more detail procedures description for the MoM methods for open-ended coaxial probe was available in following section.

2. ANALYTICAL METHOD

In this work, the derivations of analytical formulations are not shown explicitly, since that information was available in the literatures [1–5]. For coaxial probe, the scattering aperture analysis was widely referred to axisymmetric transverse magnetic (TM) formulation as [2, 4]:

$$H_\phi^+ = \frac{j\omega\varepsilon_o\varepsilon_r}{\pi}\int_a^b\int_0^\pi E_o\frac{\exp(-jkr)}{r}\cos\phi'\rho' d\phi' d\rho' \tag{1}$$

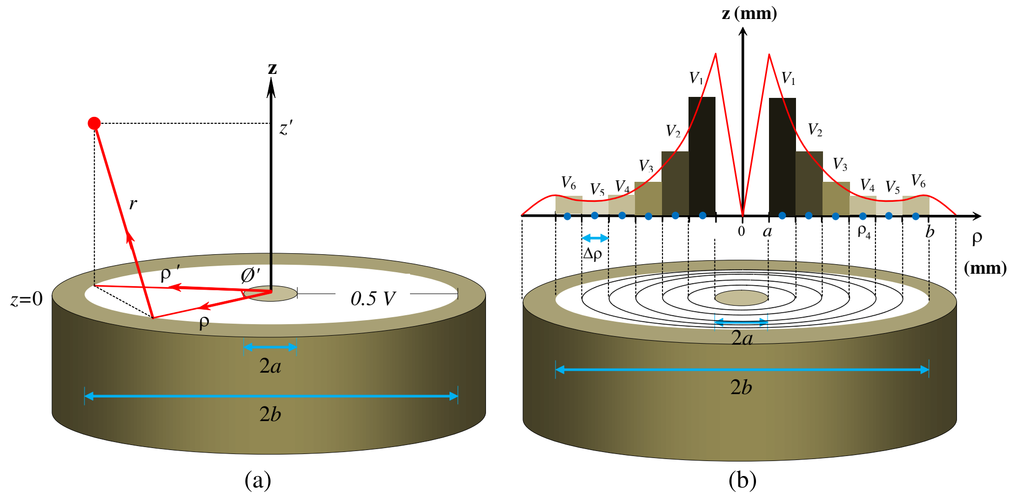

Figure 1: (a) Cylindrical coordinates used in evaluating the aperture probe radiation field analytically. (b) Vector voltage V_n for each grid positions at surface of the aperture probe.

where E_o is the magnitude of radial electric field. The distance between two points, r, at exterior region as shown in Figure 1(a) is written as:

$$r = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi' - \phi) + (z - z')^2} \tag{2}$$

The ω and k are the angular frequency and propagation constant for the external medium at exterior region (at $z' \geq 0$), respectively. Symbols of ε_o and ε_r are the permittivity of free space and relative permittivity of the external medium (at $z' \geq 0$), respectively. At discontinuity aperture, $z = z' = 0$, the magnitude of radial electric field, E_o in (1) is given as:

$$E_o = \frac{V_o}{\rho' \ln(b/a)} \tag{3}$$

where V_o is the voltage between the inner and outer conductors ($a < \rho < b$) at the aperture. For the $V_o = 1/2\,\mathrm{V}$ excitation, the (1) is re-written as [2]:

$$H_\phi^+ = \frac{j\omega\varepsilon_o\varepsilon_r}{2\pi \ln(b/a)} \int_a^b \int_0^\pi \frac{\exp(-jkr)}{r} \cos\phi' d\phi' d\rho' \tag{4}$$

From (4), the integral formulation for E_ρ component at exterior medium ($z' > 0$) is given as [2]:

$$E_\rho = \frac{-1}{j\omega\varepsilon_o\varepsilon_r} \frac{\partial H_\phi}{\partial z} = \frac{-1}{2\pi \ln(b/a)} \int_a^b \int_0^\pi \frac{\exp(-jkr)}{r} \cos\phi' \left(jk + \frac{1}{r}\right)(z - z')\, d\phi' d\rho' \tag{5}$$

The aperture admittance integral equations for open-ended coaxial radiator can be derived by matching the field components, $H_\phi^- = \frac{I}{2\pi\rho}$ (inner region, $z \leq 0$) and (4) (exterior region, z ≥ 0), since those tangential fields are continuous across the interface aperture ($z = z' = 0$):

$$H_\phi^- = H_\phi^+ \tag{6a}$$

$$\frac{I}{2\pi\rho} = \frac{j\omega\varepsilon_o\varepsilon_r}{\pi} \int_a^b \int_0^\pi E_o \frac{\exp(-jkr)}{r} \cos\phi' \rho' d\phi' d\rho' \tag{6b}$$

The current, I in (6b) are substituted by the following relationship: $I = VY$ and $V = \int_a^b E_o\, d\rho$, yields

$$\frac{Y}{2\pi\rho} \int_a^b E_o\, d\rho = \frac{j\omega\varepsilon_o\varepsilon_r}{\pi} \int_a^b \int_0^\pi E_o \frac{\exp(-jkr)}{r} \cos\phi' \rho' d\phi' d\rho' \tag{7}$$

where Y is the input admittance on surface, $z = z' = 0$. After some mathematical operation work, the input admittance, Y, which is normalized by characteristic admittance, Y_o, of coaxial line can be formulated. There are several expression form of normalized admittance, $\tilde{Y}_{in}$ are tabulated in Table 1.

Table 1: Various types of input admittance formulations of open-ended coaxial probe available in literatures.

Normalized Admittance Formulations of Open-Ended Coaxial Probe (only TEM mode)	
$\tilde{Y}_{in} = \frac{jk\varepsilon_r}{\sqrt{\varepsilon_c}\ln(b/a)} \int_0^\infty \frac{d\zeta}{\zeta(\zeta^2 - k^2)^{1/2}} \left\|J_o(\zeta a) - J_o(\zeta b)\right\|^2$ [1]	(8)
$\tilde{Y}_{in} = \frac{jk\varepsilon_r}{\pi\sqrt{\varepsilon_c}\ln(b/a)} \int_a^b \int_a^b \int_0^\pi \frac{\exp(-jkr)}{r} \cos\phi' d\phi' d\rho' d\rho$ [5]	(9)
$\tilde{Y}_{in} = \frac{1}{\eta_c \ln(b/a)} \int_a^b \frac{1}{\rho'} d\rho' + \frac{j\omega\varepsilon_o\varepsilon_r\rho}{2\pi} \int_a^b \int_{-\pi}^{\pi} \frac{\exp(-jkr)}{r} \cos\phi' \rho' d\phi' d\rho'$ [4]	(10)
$\tilde{Y}_{in} = \frac{2V_o}{\int_a^b E_\rho d\rho'} - 1$ [4]	(11)

3. NUMERICAL METHOD (MOMENT OF METHOD)

In method of moment (MoM) solution, the analytical integrals in Section 2 have been discretized and expressed it in matrix form. In fact, the MoM basically involves 3 main steps:

1. Derive an unknown parameter in the algebraic equations using Ohm's law ($V = IZ$ or $VY = I$) and analytical integral (The unknown parameter normally either is vector current, I or vector voltage, V).
2. Solve the unknown parameter using matrix routine (such as Gaussian elimination method).
3. Interpret the results in desired domains. (In H field domain, E field domain, input impedance, Z_{in} or input admittance, Y_{in}).

For coaxial probe case, the vector voltage, V_n at the aperture probe surface is assumed as the unknown parameter and the algebraic equations to be solved which is written in matrix form as [4]:

$$[Y_{mn}][V_n] = [I_m] \tag{12}$$

where Y_{mn} and I_m are the admittance and the vector current at surface of the aperture probe ($z = z' = 0$), respectively [4]:

$$Y_{mn} = \frac{\ln\left(\frac{\rho_n + \Delta\rho/2}{\rho_n - \Delta\rho/2}\right)}{\eta_c \ln(b/a)} + \frac{j\omega\varepsilon_o\varepsilon_r\rho_m}{2\pi} \int_{\rho_n - \Delta\rho/2}^{\rho_n + \Delta\rho/2} \int_{-\pi}^{\pi} \cos\phi' \frac{\exp\left(-jk\sqrt{\rho_m^2 + \rho'^2 - 2\rho_m\rho'\cos\phi'}\right)}{\sqrt{\rho_m^2 + \rho'^2 - 2\rho_m\rho'\cos\phi'}} d\phi' d\rho' \tag{13}$$

and

$$I_m = \frac{2V_o}{\eta_c \ln(b/a)} \tag{14}$$

where η_c is the intrinsic impedance of the coaxial line. The ε_o and ε_r are the permittivity of free space and the relative permittivity of the exterior medium, respectively. The unknown vector voltages, $[V_n]$ on aperture probe are determined by inverting the admittance matrix, $[Y_{mn}]$ and multiplying with the, $[I_m]$. The radial electric field, $E_{\rho n}$ at n match point position on surface aperture, can be found by substituting the set values of $[V_n]$ into (15):

$$E_{\rho n} = \begin{cases} \frac{V_n}{\rho} & \text{for } \left(\rho_n - \frac{\Delta\rho}{2}\right) < \rho < \left(\rho_n + \frac{\Delta\rho}{2}\right) \\ 0 & \text{other} \end{cases} \tag{15}$$

Once the electric field, $E_{\rho n}$ on each match point positions has been determined, the (1) can be re-exam by replacing (15) into (1) as:

$$H_\phi^+ = \frac{j\omega\varepsilon_o\varepsilon_r}{\pi} \sum_{n=1}^{N} V_n \int_{\rho_n - \Delta\rho/2}^{\rho_n + \Delta\rho/2} \int_0^\pi \frac{\exp(-jkr)}{r} \cos\phi' d\phi' d\rho' \tag{16}$$

Besides that, from the total value of electric field, $\sum V_n$, the input admittance, $\tilde{Y}_{in}$ of the entire surface aperture can be calculated by discretizing the integral electric field term in (11) with $V_0 = 1\,\mathrm{V}$, yields:

$$\tilde{Y}_{in} = \left(\frac{2V_o}{\sum\limits_{n=1}^{N} V_n \ln\left(\frac{\rho_n + \Delta\rho/2}{\rho_n - \Delta\rho/2}\right)} - 1 \right) \tag{17}$$

4. RESULTS AND DISCUSSION

In this work, the vector voltage, V_n on surface of aperture coaxial probe was simulated using $N = 10$ match points along the radius, ρ as depicted in Figure 2(a). The calculations assumed that the section of coaxial line ($a < \rho < b$) is filled with Teflon ($\varepsilon_c = 2.06$) and the value ratio of $b/a = 3.298$ with $a = 0.4545\,\mathrm{mm}$. In addition, the external dissipative medium at exterior region ($z' > 0$) was assumed to be methanol liquid and its dispersive properties obtained from Cole-Cole model with $\varepsilon_s = 33.7$, $\varepsilon_\infty = 4.35$, $\tau = 49.64\,\mathrm{ps}$ and $\alpha = 0.0426$.

From Figure 2(b), the radial electric field decay exponentially as expected with the radial distance ($E_\rho \propto 1/\rho$). The comparison between analytical and numerical results for the magnetic field, $H_\emptyset$ and electric field, E_ρ at 3 GHz are shown in Figures 3(a) and (b), respectively. The absolute values of $H_\emptyset$ and E_ρ were normalized with its own maximum values ($H_{\emptyset\,\max}$ and $E_{\rho\,\max}$) to facilitate the comparison. The calculation for $H_\emptyset$ and E_ρ using (4) and (5), respectively, were conducted by a 12×12-order Gaussian double integral method. Figure 3(a) shows the limitation of analytical calculation using (4) at aperture probe ($z = z' = 0$) and poor agreement with the numerical calcu-

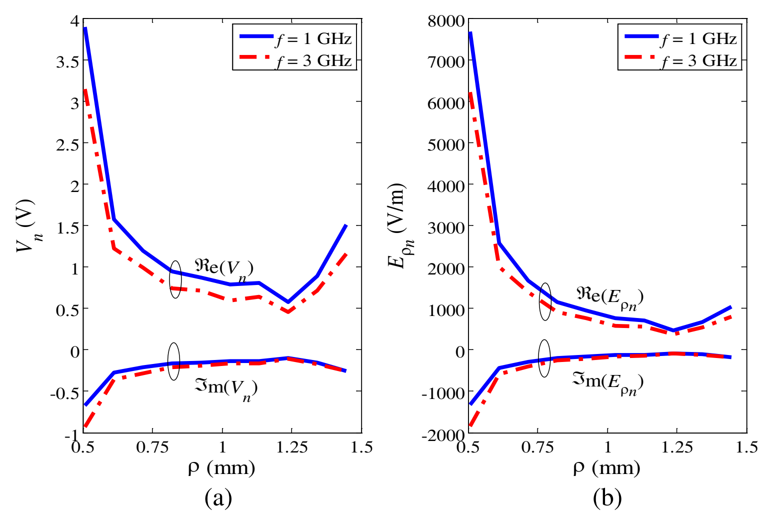

Figure 2: (a) The vector voltage, V_n and (b) vector electric field, $E_{\rho n}$ along the radius, ρ axis on surface aperture ($z = z' = 0$).

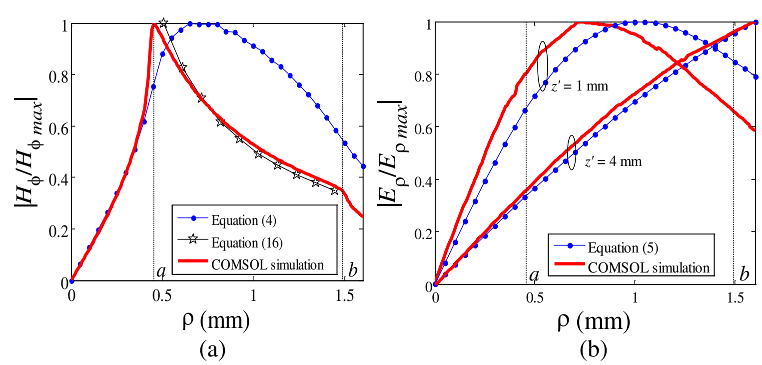

Figure 3: (a) The normalized absolute magnetic field versus radius, ρ axis on surface aperture ($z' = 0$). (b) The normalized absolute electric field versus radius, ρ on $z' = 1\,\mathrm{mm}$ plane and $z' = 4\,\mathrm{mm}$ plane (above the surface aperture).

lations [Equation (16) and COMSOL simulation]. From Figure 3(b), the analytical calculation of E_ρ using (5) was closed to the COMSOL simulation results when the observation plane moves far away from the aperture. It deduced that the Equation (5) is only suitable for far field observation.

The normalized input admittance, $\tilde{Y}_{in}$, is a complex number which can be separated into normalized values of conductance, $G(0)/Y_o$ and susceptance, $B(0)/Y_o$ as $\tilde{Y}_{in} = G(0)/Y_o + jB(0)/Y_o$. The comparison of calculated $G(0)/Y_o$ and $B(0)/Y_o$ using models [(8), (9), (17), and COMSOL simulation], and the simulation results for methanol are showed in Figures 4(a) and (b), respectively. The calculation for admittance in Equation (9) was conducted by a $5 \times 5 \times 7$-order Gaussian triple integral method, while Equation (8) was solved by the series expansion method with 9 terms of series [6]. Figures 4(a) and (b) clearly show good agreement for the $G(0)/Y_o$ and $B(0)/Y_o$ results which are obtained from Equation (17) and COMSOL simulation. The ignoring of TM_{0n} modes in (8) and (9) causes low accuracy of the analytical calculation results; however, it leads to a relatively simple mathematical solution.

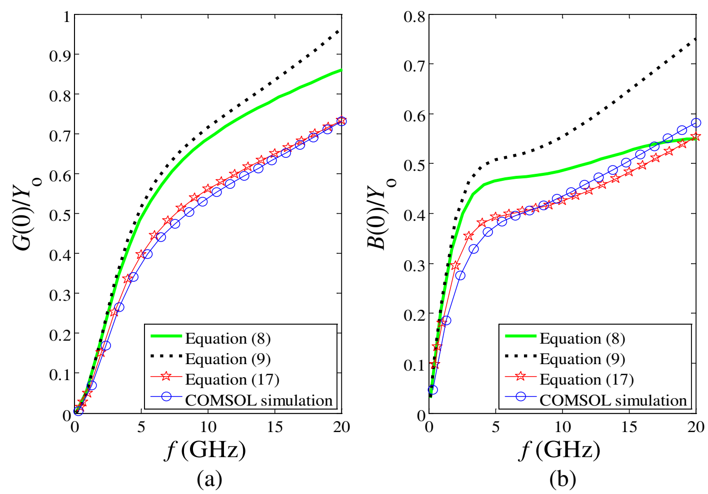

Figure 4: The calculated (a) normalized conductance, $G(0)/Y_o$ and (b) normalized suscepatnce $B(0)/Y_o$ for methanol at 25°C by considering size of probe with $b/a = 3.298$ and $a = 0.4545$ mm.

5. CONCLUSION

In this work, the coaxial probe has been used as an example of the electromagnetic field analysis, in order to link the analytical formulations with numerical solutions. The restrictions and limitations of analytical models for coaxial probe have been studied and the reasons for replacement of analytical method by a numerical method have been briefly discussed.

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