# Effect of a Linear Frequency Modulation on the Nonlinear Dynamics of an Electromagnetic Pulse in a Graded-index Waveguide

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**Abstract**— The present paper is aimed at a consistent analytical description of a combined interplay of linear frequency modulation of the high-frequency carrier, nonlinearity, and inhomogeneity of an optical fiber on the propagation of an arbitrary mode feasible in the fiber, with vortex modes included. Formal asymptotic procedure is applied to the problem of wave propagation and equations for the mode structure and the envelope of the chirped pulse are derived. Lagrangian is proposed enabling to reformulate the envelope equation as a variational problem which is then solved for the case of travelling pulse of an arbitrary shape. Explicit expressions for the pulse parameters are derived via characteristics of the gradient optical fiber and the structure of propagating mode.

## 1. INTRODUCTION

Since the prediction of bright and dark optical solitons in media with proper dispersive characteristics [1] and the experimental observation of soliton formation in optical fibers [2] a great progress has been achieved in sophistication and application of nonlinear localized optical waves in various artificial structures [3]. Optical fibers turned out to be a very favourable and common medium due to a low threshold of nonlinear effects excitation [4] and feasibility of waveguiding devices with stable geometry. Various types of optical fibers are now being produced commercially — mono- and multimode, stepwise and graded-index etc. — with each of them having its utilization area. Design of specific refractive index profiles enables to control the dispersive characteristics of optical waveguides. Use of a linear frequency modulation can partly mitigate the dispersive broadening, so far as, with the proper sign of the linear frequency modulation being chosen, the modulation overplays the dispersion, which leads to pulse compression up to a certain propagation distance, and only on passing this distance the pulse dispersive spreading becomes irreversible. An analytical account of nonlinearity in propagation of a pulse with a quadratic phase modulation in a graded-index optical fiber has been recently attained by means of an asymptotic procedure [5] with respect to the pulse amplitude assumed to be a small parameter. Stated in [5] is the necessity to discern between pulses with modulation depth of different orders of magnitude, and the nonlinear Schrödinger equation for the pulse envelope is generalized onto the case where phase modulation and longitudinal inhomogeneity of the waveguide are taken into account.

So far as the information capacity of any transmission system is limited with a finite bandwidth and noise [6], a point is once attained at which only novel physical ideas or technologies could provide a further growth in the information transmission rate. Recent developments in fiber optics communications are now toughly coming at the “capacity crunch” conditioned by approaching the fundamental limits due to signal-to-noise ratio and nonlinearity of available optical fibers [7]. A possible way to provide a principal capacity growth consists in utilization of the mode division multiplexing in fibers, and involvement of the modes with orbital angular momentum [8]. Reported in [9] is a phase diffractive optical element and algorithm forming an informative set of optical vortices in a laser beam in the fiber. The present paper is aimed at a consistent analytical description of a combined interplay of linear frequency modulation of the high-frequency carrier, nonlinearity, and inhomogeneity, either transverse and longitudinal, of the fiber, on the propagation of an arbitrary mode feasible in the fiber, with vortex modes included. A variational approach stemming from Whitham's principle [10] is used in the analysis of the equation derived for the pulse envelope, this succeeded in determination of practical characteristics of the short optical pulse.

## 2. NONLINEAR WAVE EQUATION AND ANSATZ

The short pulse propagation in a graded-index waveguide is expedient to be treated as a weak-nonlinear process thus enabling to combine modal structure of the pulse and its nonlinear evolution along the waveguide axis. Thus a small parameter $\delta$ which should be introduced for asymptotic analysis is chosen as an order of magnitude of the pulse amplitude. Henceforth dimensionless variables are used: $\rho$ — radial coordinate multiplied by the wavenumber *in vacuo*, $s$ — longitudinal coordinate multiplied by the wavenumber *in vacuo* and $\delta^2$ (i.e., squeezed proportionally to $\delta^2$), $t$ — time multiplied by the angular frequency, thus the nonlinear wave equation is written down as follows

$$\Delta f - \left(\beta^2(\rho,s) + \frac{1}{2}\alpha^2(\rho,s)\,|f|^2\right)\frac{\partial^2 f}{\partial t^2} = 0 \tag{1}$$

$\alpha$, the Kerr coefficient, characterizes the nonlinear properties of the medium, $\beta$ defines the refractive index profile within the cross-section in the linear regime of propagation, accounting for a longitudinal inhomogeneity via dependence on $s$. The solution to Equation (1) is sought for as

$$f(\rho,\varphi,s,t) = \delta F(\rho,\varphi,s,\theta)\exp\left[i\left(R(s)/\delta^2 - t - \delta^3\mu(s)t^2\right)\right] + \text{c.c.} \tag{2}$$

$$\theta(s,t) = \frac{1}{\delta}Q(s) - \delta t \qquad R(s) = \int_0^s r(s')\mathrm{d}s' \qquad Q(s) = \int_0^s q(s')\mathrm{d}s'$$

the linear frequency modulation depth is taken as $\delta^3\mu(s)$ admitting its dependence along the propagation path and assuming its order of magnitude corresponding to ordinary modulation [5]. An essential feature of the Ansatz (2) consists in introduction of the envelope phase $\theta$, so the instantaneous wavenumbers $r(s)$ of the high-frequency carrier and $q(s)$ of the envelope are supposed to be different functions on $s$. The complex amplitude is represented as a series $F(\rho,\phi,s,\theta) = \sum_{j=0}^{\infty}\delta^j F_j(\rho,\phi,s,\theta)$ in powers of the small parameter. We'll require that the complex amplitude vanishes with distancing from the waveguide axis.

## 3. MODE STRUCTURE AND ENVELOPE OF THE CHIRPED PULSE

The main order complex amplitude obeys the equation

$$\frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \varphi^2} + \left(\beta^2(\rho,s) - r^2(s)\right)F = 0 \tag{3}$$

and must be $2\pi$-periodic with respect to $\varphi$, bounded at the axis $\rho = 0$, and vanish for $\rho \to \infty$. The azimuthal dependence could be chosen via either $\{\cos m\phi,\ \sin m\phi\}$ or $\{e^{\pm im\phi}\}$, the latter corresponds to a vortex mode with a screw-like wave front. The structure of the problem in (3) implies the possibility to present a vortex mode as a product $F(\rho,\phi,s,\theta) = V(\rho,s)U(s,\theta)e^{im\phi}$, where $V(\rho,s)$ the normalized eigenfunction of the Sturm-Liouville problem reduced from (3) by extracting the azimuthal dependence ($r(s)$ is the eigenvalue) and $U(s,\theta)$ characterizes the dynamics of the pulse envelope. Further approximations to the complex amplitude $F$ are determined from inhomogeneous equations with left-hand side analogous to that in (3) and with right-hand side calculated in previous steps of the asymptotic procedure. In order to obey the boundary conditions for $F_j$ one has to impose conditions of compatibility onto the right-hand sides, which leads to an additional series of equations. So the problem on $F_1$ allows to obtain an expression

$$q(s) = r(s) + \frac{1}{r(s)}\int_0^\infty \rho\left(\frac{\partial V}{\partial \rho}\right)^2\mathrm{d}\rho + \frac{m^2}{r(s)}\int_0^\infty \frac{V^2}{\rho}\mathrm{d}\rho \tag{4}$$

which, according to (2), determines the phase of the pulse envelope, depending on the order of vorticity $m$. Another inference consists in stating the relationship

$$\mu(s)Q^2(s) = \mu_0 Q_0^2 = \text{const} \tag{5}$$

meaning that the carrier modulation depth evolution is strictly bound to the envelope phase. The compatibility condition for the problem on $F_2$ results in a nonlinear equation describing the dynamics of the pulse envelope

$$2ir(s)\frac{\partial U}{\partial s} + g(s)\frac{\partial^2 U}{\partial \theta^2} + ij(s)\frac{\partial U}{\partial \theta} + ir'(s)U + 4\theta\mu(s)r(s)q(s)U + d(s)U + h(s)\,|U|^2\,U = 0 \tag{6}$$

The pulse envelope $U$ is a function of the envelope phase $\theta$ defined by (2) and longitudinal coordinate $s$, $r(s)$ is defined as well by (2), and $r'(s)$ is the derivative. Formulae for coefficients $g(s)$, $j(s)$,

$d(s)$ and $h(s)$ are derived quite analogously to those in [5], their dependence on $s$ is caused by the longitudinal inhomogeneity of the waveguide. It should be emphasized that these coefficients are specific for a mode chosen in solving (3), henceforth Equation (6) is valid for any radial, azimuthal or vortex mode. Explicit formulae can be readily obtained for a quadratic dependence of the refractive index on the radial coordinate.

## 4. VARIATIONAL APPROACH TO ENVELOPE ANALYSIS

The equation for the pulse's envelope (6) can be analyzed by means of variational approach [11, 12] which had been substantiated in [13] for generalized nonlinear Schrödinger equation with additional terms of some special kind. Note first of all that Equation (6) turns out to be an Euler equation for the following Lagrangian density

$$\begin{aligned}\mathcal{L}(s,\theta) \;=\; & ir(s)\left(\frac{\partial U}{\partial s}U^* - U\frac{\partial U^*}{\partial s}\right) - g(s)\frac{\partial U}{\partial \theta}\frac{\partial U^*}{\partial \theta} + d(s)UU^* \\ & + \frac{1}{2}ij(s)\left(\frac{\partial U}{\partial \theta}U^* - U\frac{\partial U^*}{\partial \theta}\right) + 4\theta\mu(s)r(s)q(s)UU^* + \frac{1}{2}h(s)\left(UU^*\right)^2 \end{aligned} \tag{7}$$

One can try to find a solution in the form of a travelling wave of unknown shape $P(x)$ with the variable amplitude $A(s)$, phase $\Phi(s)$, instant envelope frequency $\Omega(s)$, width $\sigma(s)$ and the envelope shift $\Delta(s)$

$$U(s,\theta) \;=\; A(s)P(x)e^{i\Omega(s)\theta + i\Phi(s)} \qquad x(s,\theta) \equiv \frac{\theta - \Delta(s)}{\sigma(s)} \tag{8}$$

$$U(0,\theta) \;=\; A_0P(x_0)e^{i\Omega_0\theta + i\Phi_0} \qquad x_0 \equiv \frac{\theta - \Delta_0}{\sigma_0} \tag{9}$$

Substituting these into the Lagrangian density (7) we come to

$$\mathcal{L}_P(s,\theta) = -2r\left(\frac{\mathrm{d}\Omega}{\mathrm{d}s}\theta + \frac{\mathrm{d}\Phi}{\mathrm{d}s}\right)A^2P^2 - \frac{g}{\sigma^2}A^2\left(\frac{\mathrm{d}P}{\mathrm{d}x}\right)^2 + \frac{1}{2}hA^4P^4 - \left(g\Omega^2 + j\Omega - d\right)A^2P^2 + 4\theta\mu rqA^2P^2$$

We can now integrate the Lagrangian density $\mathcal{L}_P$ over the phase variable $\theta$ resulting in the reduced Lagrangian

$$\begin{aligned}\left\langle\mathcal{L}_P\right\rangle(s) \;\equiv\; & \int_{-\infty}^{+\infty}\mathcal{L}_P(s,\theta)\mathrm{d}\theta = -\frac{\sigma(s)}{c(s)}\int_{-\infty}^{+\infty}\mathcal{L}_P(s,x)\mathrm{d}x \\ =\; & -2r\sigma\Delta\left(\frac{\mathrm{d}\Omega}{\mathrm{d}s} - 2\mu q\right)AI_0 - 2r\sigma^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}s} - 2\mu q\right)AI_1 - 2r\sigma\frac{\mathrm{d}\Phi}{\mathrm{d}s}A^2I_0 \\ & -2\frac{g}{\sigma}AI_2 - \sigma(g\Omega^2 + j\Omega - d)AI_0 + \frac{1}{2}h\sigma A^4I_3 \end{aligned} \tag{10}$$

where the following integrals are introduced

$$I_0 = \int_{-\infty}^{+\infty}P^2(x)\mathrm{d}x \qquad I_1 = \int_{-\infty}^{+\infty}xP^2(x)\mathrm{d}x \qquad I_2 = \int_{-\infty}^{+\infty}\left(\frac{\mathrm{d}P}{\mathrm{d}x}\right)^2\mathrm{d}x \qquad I_3 = \int_{-\infty}^{+\infty}P^4(x)\mathrm{d}x$$

For the reduced Lagrangian (10) then follows the principle of the least action $\delta\int\left\langle\mathcal{L}_P\right\rangle\mathrm{d}s = 0$. It allows us to determine the optimal functional dependencies for the pulse parameters. By taking functional derivatives with respect to the pulse parameters $A$, $\sigma$, $\Delta$, $\Omega$ and $\Phi$ one can, after some manipulations, obtain the following explicit expressions

$$A^2(s) \;=\; \frac{D^2}{4}\frac{I_3}{I_2}\frac{h}{gr^2} \tag{11}$$

$$\Omega(s) \;=\; \Omega_\infty - 2\frac{\mu_0Q_0^2}{Q(s)} \tag{12}$$

$$\sigma(s) \;=\; \frac{4}{D}\frac{I_2}{I_3}\frac{g(s)r(s)}{h(s)} \tag{13}$$

where $D \equiv r(0)\sigma_0 A_0^2$ and $\Omega_\infty \equiv \Omega_0 + 2\mu_0 Q_0$. The envelope shift $\Delta(s)$ and envelope phase $\Phi(s)$ in (8), (9) are expressed as follows

$$\Delta(s) = \Delta_0 + \Omega_\infty \int_0^s \frac{g(s')}{r(s')} \mathrm{d}s' + \frac{I_1}{I_0}\left(\sigma_0 - \sigma(s)\right)$$

$$\Phi(s) = \Phi_0 + \int_0^s \left[\frac{3D}{8}\frac{h(s')}{r^2(s')} - \frac{1}{2r(s')}\left(\Omega_\infty^2 g(s') - 4\frac{\mu_0 \Omega_0^2}{Q^4(s')} g(s') - d(s')\right)\right] \mathrm{d}s'$$

To illustrate the results of this analysis one can look at the simple case of pulse propagation in an optical fiber with a refractive index profile that remains constant along the fiber's length. In this case the Equations (5), (11), (12), and (13) can be simplified leading to following explicit expressions

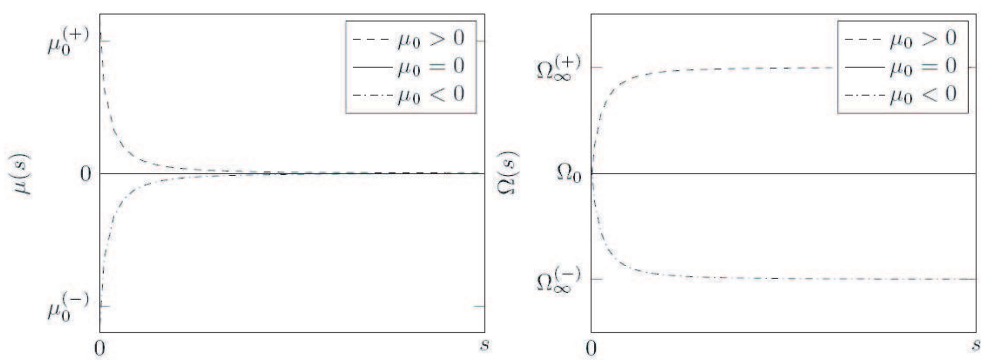
$$A^2(s) = A_0^2 \tag{14}$$

$$\sigma(s) = \sigma_0 \tag{15}$$

$$\mu(s) = \frac{\mu_0 Q_0^2}{\left(Q_0 + qs\right)^2} \tag{16}$$

$$\Omega(s) = \Omega_\infty - 2\frac{\mu_0 Q_0^2}{(Q_0 + qs)} \tag{17}$$

From (14) and (15) it follows, that amplitude and width of the pulse remain constant along the fiber's length. Qualitatively dependencies (16) and (17) are sketched in the Figure 1. It can be seen that while the pulse propagates the phase modulation depth $\mu(s)$ slowly decays to zero, thus leveling the instant carrier frequencies of the front and the rear ends of the pulse. This effect is accompanied by the envelope frequency $\Omega(s)$ asymptotically approaching it's final value of $\Omega_\infty$ that is either greater or less than initial frequency $\Omega_0$ depending on the sign of the $\mu_0$.




(a) (b)

Figure 1: Qualitative asymptotic behaviour of the (a) chirp depth $\mu(s)$ and (b) envelope frequency $\Omega(s)$ in a fiber with a profile independent on $s$. Constants $\mu_0^{(-)}$ and $\Omega_0^{(-)}$ correspond to the case of $\mu_0 < 0$. Constants $\mu_0^{(+)}$ and $\Omega_0^{(+)}$ correspond to the opposite case of $\mu_0 > 0$.

## 5. CONCLUSION

An outline has been presented for the asymptotic procedure needed to describe propagation of an arbitrary, vortex included, mode with linear frequency modulation of the carrier in a nonlinear graded-index optical fiber with slowly-varying longitude inhomogeneity. The Lagrangian density for the envelope equation was guessed and the problem was analyzed by means of a variational approach. The explicit expressions for the pulse amplitude, width, phase shift, envelope shift and envelope frequency were obtained for an arbitrary propagating mode. The modulation depth is found to be decaying along the fiber's length, while the envelope frequency tends asymptotically to some specific value that depends on the initial modulation frequency and depth.

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