# Aharonov-Bohm Effect, Poincaré Lemma and Gauge Invariance

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**Abstract**— In this paper we demonstrate the peculiarities of the Aharonov-Bohm effect using the theory of generalized functions. We show that the presence of the Aharonov-Bohm effect violates the Poincaré lemma (in other words the Poincaré lemma does not apply to the Aharonov-Bohm spatial geometry), moreover, such a violation is a necessary condition for the Aharonov-Bohm effect observation. The unexpected form of the Schrödinger equation for the Aharonov-Bohm effect which contains a singularity in explicit form is also obtained. These results allow to take a fresh look on the Aharonov-Bohm problem.

## 1. INTRODUCTION

The Aharonov-Bohm (AB) effect [1] is of more than fifty years history and remains relevant even in the present days (see, e.g., [2–7] and related references). Main object in the AB effect is an infinitely thin solenoid with the finite magnetic flux (AB flux). While the force field is absent the vector potential of the magnetic field confined in the solenoid is non-zero and can produce observable effects because the relative phase of the electron wave function depends on the AB flux. In other words, the AB flux affects the quantum state of an electron. Such an affection leads eventually to the observable effects [8, 9].

As is known, there is a close "relationship" between the AB effect and the topology of the two-dimensional space (see, e.g., [10]). One of the most important result in this field is the Poincaré lemma [11, 12] which claims that on a contractible manifold, all closed forms are exact. In other words $\mathbf{dd}\omega = 0$ for any closed form $\omega$. In this paper we show by straightforward calculations that in the presence of the AB effect the Poincaré lemma violates (the Poincaré lemma can not be applied to the AB problem due to the fact that the space is not simply-connected). Such a violation connected with the fact that the 1-form $\mathbf{d}\omega$ (see details below in Eq. (11)) is not exact.

The paper is organized as follows. In the first section we demonstrate (using the theory of generalized functions) the significant difference between the gauges of the homogeneous magnetic field and AB field. The second section is dedicated to the problem of Poincaré lemma violation. We show by the straightforward calculations that the presence of the AB solenoid leads to Poincaré lemma violation. In conclusion the main results of this paper are summarized.

## 2. AHARONOV-BOHM GAUGE AND MULTI-VALUED FUNCTION

In this section we would like to show the specific difference between the gauge of the vector potential in the case of the AB flux and homogeneous magnetic field. As is well known we can use the various form of the gauge for the vector potential. The most convenient is the following expression for the vector potential of a uniform magnetic field:

$$\mathbf{A}_H = \frac{Hr}{2}\mathbf{e}_\theta. \tag{1}$$

Formally, we can write

$$\mathbf{A}_H = \frac{\Phi_H}{2\pi r}\mathbf{e}_\theta, \tag{2}$$

where $\Phi_H = \pi r^2 H$ is the magnetic flux through the area of the ring with radius $r$.

For the vector potential of the AB solenoid (with the AB flux $\Phi_{AB}$) we have [1]:

$$\mathbf{A}_{AB} = \frac{\Phi_{AB}}{2\pi r}\mathbf{e}_\theta. \tag{3}$$

Let us note that the AB vector potential (3) is singular at origin $r = 0$. At the same time the vector potential of the homogeneous magnetic field (see the formal expression (2)) is zero at origin due to the fact that the magnetic flux $\Phi_H$ is a quadratic function of $r$.

Now we can pose the following question: Does the gauge transformation (non-singular) which connects (3) and (2) exist? The answer is: no, it does not. Further calculations confirm this response.

First, we write a well known expression for the operator $\nabla$ in cylindrical coordinates:

$$\nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e}_z. \tag{4}$$

Next, for the homogeneous magnetic field (1) we have

$$[\nabla \times \mathbf{A}_H] = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\mathbf{A}_H\right)_\theta\right]\mathbf{e}_z = H\mathbf{e}_z. \tag{5}$$

The situation dramatically changes when we consider the case of the AB flux. Using (4) we can rewrite (3) as [13]:

$$\mathbf{A}_{AB} = \frac{\Phi_{AB}}{2\pi}\nabla\theta. \tag{6}$$

Then, we have

$$[\nabla \times \mathbf{A}_{AB}] = \frac{\Phi_{AB}}{2\pi}[\nabla \times \nabla\theta].$$

Now we show that the expression $[\nabla \times \nabla\theta]$ is nonzero and reduces to a Dirac $\delta$-function (this occurs because $\theta$ is a multi-valued function). In the 2D case we can write:

$$[\nabla \times \nabla\theta] = \left(\frac{\partial^2\theta}{\partial x \partial y} - \frac{\partial^2\theta}{\partial y \partial x}\right)\mathbf{e}_z.$$

For the azimuthal angle we know:

$$\theta = \arctan\left(\frac{y}{x}\right),$$

and

$$\begin{aligned}\frac{\partial\theta}{\partial x} &= -\frac{y}{x^2+y^2},\\ \frac{\partial\theta}{\partial y} &= \quad\frac{x}{x^2+y^2}.\end{aligned}$$

Note, that the functions $\frac{\partial\theta}{\partial x}$ and $\frac{\partial\theta}{\partial y}$ are functions with the homogeneity parameter $\lambda = -1$.

Now, let us recall some properties of the homogeneous functions. The homogeneity parameter $\lambda$ is defined as follows:

$$f(\xi x_1, \xi x_2, \ldots, \xi x_n) = \xi^\lambda f(x_1, x_2, \ldots, x_n)$$

for all $\xi > 0$. In our case $\lambda = -n + 1 = -1$ (here $n = 2$ is a space dimension of the system under consideration). Thus, for such a function we need to use the following expression for the derivative [14]:

$$\frac{\partial f}{\partial x_i} = \left.\left(\frac{\partial f}{\partial x_i}\right)\right|_G + (-1)^{i-1}\delta\left(x_1, x_2, \ldots, x_n\right) \times \int\limits_\Gamma f\, dx_1 dx_2 \ldots dx_{i-1} dx_{i+1} \ldots dx_n. \tag{7}$$

Here $\frac{\partial f}{\partial x_i}$ is a derivative in terms of generalized functions, and $\frac{\partial f}{\partial x_i}\left.\right|_G$ is a generalized function, built on the usual derivative of $f$ in the region $G$, $\Gamma$ is the boundary of $G$, $\delta\left(x_1, x_2, \ldots, x_n\right)$ is a multidimensional Dirac $\delta$-function. Next, in our case using (7) we obtain

$$\begin{aligned}\frac{\partial^2\theta}{\partial x \partial y} &= \left.\frac{\partial^2\theta}{\partial x \partial y}\right|_G + \delta(x, y)\int\limits_\Gamma \frac{x dy}{x^2+y^2},\\ \frac{\partial^2\theta}{\partial y \partial x} &= \left.\frac{\partial^2\theta}{\partial y \partial x}\right|_G + \delta(x, y)\int\limits_\Gamma \frac{y dx}{x^2+y^2}.\end{aligned} \tag{8}$$

Of course, $\left.\frac{\partial^2\theta}{\partial x\partial y}\right|_G = \left.\frac{\partial^2\theta}{\partial y\partial x}\right|_G$, then using (8) we find

$$[\nabla \times \nabla\theta] = \vec{e}_z\delta(x,y)\int\limits_\Gamma \frac{xdy - ydx}{x^2+y^2},$$

and

$$[\nabla \times \mathbf{A}_{AB}] = \mathbf{e}_z\frac{\Phi_{AB}\delta(x,y)}{2\pi}\int\limits_\Gamma \frac{xdy - ydx}{x^2+y^2}.$$

Note, that in the present case $\Gamma$ can be chosen (in full analogy with an example considered in [14]), e.g., as a circle, so the contour integral can be easily calculated (in the polar coordinates) and the result is

$$\int\limits_\Gamma \frac{xdy - ydx}{x^2+y^2} = 2\pi.$$

Finally

$$[\nabla \times \mathbf{A}_{AB}] = \Phi_{AB}\mathbf{e}_z\delta(x,y). \tag{9}$$

The expressions (5) and (9) demonstrate a significant difference between the uniform magnetic field and the AB flux placed at origin (one of them is regular while another is singular)[1]. Thereby, the definition of the gauge invariance of the electromagnetic potentials should be clarified, namely:

$$\mathbf{A} \to \mathbf{A} + \nabla f,$$

$$\varphi \to \varphi - \frac{\partial f}{\partial t},$$

where $f$ is an arbitrary ***single-valued*** function of the spatial coordinates and time.

## 3. POINCARÉ LEMMA VIOLATION

Let us recall the Poincaré lemma: *On a contractible manifold all closed forms are exact.* In other words $\mathbf{dd}\omega = 0$ for any closed form $\omega$[2]. In this section we show that the Poincaré lemma violation takes place in the presence of the AB effect. In order to do this let us consider the 0-form $\omega = \arctan(y/x)$ that exactly corresponds to the AB gauge of the vector potential (see Eq. (6)). It should be noted one more time that the considered 0-form $\omega$ is *a multi-valued function*.

Let us expand the 1-form $\mathbf{d}\omega$ on the basis of $\mathbf{d}x$ and $\mathbf{d}y$ in the standard way [11]:

$$\mathbf{d}\omega = \frac{\partial\omega}{\partial x}\mathbf{d}x + \frac{\partial\omega}{\partial y}\mathbf{d}y,$$

and let us find the exterior derivative for this one. The result is:

$$\mathbf{dd}\omega = \frac{\partial^2\omega}{\partial x^2}\mathbf{d}x\wedge\mathbf{d}x + \frac{\partial^2\omega}{\partial y\partial x}\mathbf{d}x\wedge\mathbf{d}y + \frac{\partial^2\omega}{\partial x\partial y}\mathbf{d}y\wedge\mathbf{d}x + \frac{\partial^2\omega}{\partial y^2}\mathbf{d}y\wedge\mathbf{d}y.$$

Using the general properties of the exterior derivative [12] we can write:

$$\mathbf{d}x\wedge\mathbf{d}x = \mathbf{d}y\wedge\mathbf{d}y = 0,\ \ \mathbf{d}x\wedge\mathbf{d}y = -\mathbf{d}y\wedge\mathbf{d}x.$$

These properties lead to the following expression for the 2-form $\mathbf{dd}\omega$:

$$\mathbf{dd}\omega = \left(\frac{\partial^2\omega}{\partial y\partial x} - \frac{\partial^2\omega}{\partial x\partial y}\right)\mathbf{d}x\wedge\mathbf{d}y. \tag{10}$$

[1]It should be noted here that the numerical solution of the Newton equation (the Lorentz force) together with the field strength (9) does not lead to some observable peculiarities. This result exactly correlates with the statement on the absence of the AB effect in classical physics.

[2]Here we would like to give a simple interpretation of the Poincaré lemma, namely, the last one is equivalent to the statement that *the mixed second derivatives are equal.* For the vast majority of functions that are used in physics (single-valued functions), this statement is certainly true. However, in the case of multi-valued functions (e.g., in the AB effect) this statement can be violated. The violation of this statement is presented in this section.

The expression in the brackets in (10) is nonzero because the 0-form $\omega$ is multi-valued. As a result the second derivatives in (10) should be calculated in the frame of the theory of generalized functions [14]. Using the expression (7), we obtain the following result:

$$\mathbf{dd}\omega = -2\pi\delta(x, y)\mathbf{d}x \wedge \mathbf{d}y \tag{11}$$

We see that the Poincaré lemma violation takes place. Such a violation follows from the fact that the considered 0-form $\omega$ is a multi-valued function. In mathematical language this means that the 1-form $\mathbf{d}\omega$ is not exact. Physically this means that the singularity in the form of AB solenoid takes place (the space of the AB problem is multi-connected) and such a singularity affects the quantum state of an electron. Moreover, using the generalized functions technique such a singularity can be identified in the Schrödinger equation in explicit form.

## 4. CONCLUSIONS

In this paper we have considered the peculiarities of the AB effect in the frame of the theory of generalized functions. In particular, we have shown the significant difference between the gauges for the homogeneous magnetic field and AB field. As a result, we have pointed out that the definition of a gauge invariance should be formulated taking into account an information on the "nature" of a gauge function. Also, we have demonstrated that the Poincaré lemma violation takes place due to the presence of the AB flux. Such a violation follows from the fact that the space of the AB problem is multi-connected and the corresponding 0-form that determines the gauge of the AB field is not exact.

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