Conductor Modeling Based on Volume Integral Equations

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Abstract— The electromagnetic (EM) problems with conductive objects can be solved by surface integral equations (SIEs) with the method of moments (MoM) discretization in the integral equation approach. However, the solutions may not be valid for a wide range of frequency and conductivity. In this work, we use the volume integral equations (VIEs) to exactly formulate the problem and propose a point-matching scheme to solve it. The VIEs are usually well-conditioned because they are the second-kind of integral equation. The point-matching scheme can select current densities as unknowns so that the integral kernels are free of material parameters and the solutions can bear a significant change of frequency and conductivity. A numerical example for EM scattering by a conductive sphere with different parameters is presented to demonstrate the approach.

1. INTRODUCTION

Accurate solution of electromagnetic (EM) problems with lossy conductors requires to consider the finite conductivity of the conductors. The loss of the conductors may not be ignored when the frequency is low or the conductivity is small since the skin depth is large [1]. In the integral equation approach for solving the problem, one usually uses surface integral equation (SIE) approach with an approximate surface impedance when the skin depth is small [2]. For large skin depth caused by low frequency or small conductivity, one treats the conductors as dielectric-like objects and uses a two-region scheme to formulate the SIEs [3, 4]. The SIEs are solved by the method of moments (MoM) with the Rao-Wilton-Glisson (RWG) basis function [5]. However, the solutions may not be valid for a wide range of frequency and conductivity because the integral kernels include those parameters and the conditioning of SIEs is susceptible to the frequency [6].

In this work, we also treat the conductors as dielectric-like objects but use volume integral equations (VIEs) to exactly formulate the problem. The VIEs are the second-kind of integral equation, so they are well-conditioned in general [1]. Also, the VIEs only have one unknown function (electric current density) for dielectric (nonmagnetic) objects while two unknown functions appear in the SIEs. To solve the VIEs, we do not use the traditional MoM with the Schaubert-Wilton-Glisson (SWG) basis function [7] but propose a point-matching scheme which does not use any basis and testing functions and allows a geometric discretization of nonconforming meshes, resulting in much convenience in implementation [8]. Moreover, the scheme can choose current densities as unknowns so that the integral kernels are free of material parameters and the solutions can endure a wide change of frequency and conductivity. The scheme requires an accurate evaluation of hypersingular integrals resulting from the double gradient operation in the dyadic Green's function but we have developed a robust technique to handle them [9, 10]. We present a numerical example for EM scattering by a conductive sphere to illustrate the approach and good results can be observed.

2. VOLUME INTEGRAL EQUATIONS

Consider the EM scattering by a three-dimensional (3D) penetrable object in the free space with a permittivity ϵ_0 and a permeability μ_0, the VIEs can be written as [1]

$$\vec{E}\left(\vec{r}\right) = \vec{E}^{inc}\left(\vec{r}\right) + i\omega\mu_0 \int_V \overline{\overline{G}}\left(\vec{r},\vec{r}'\right) \cdot \vec{J}_V\left(\vec{r}'\right) d\vec{r}' - \nabla \times \int_V \overline{\overline{G}}\left(\vec{r},\vec{r}'\right) \cdot \vec{M}_V\left(\vec{r}'\right) d\vec{r}', \ \vec{r} \in V \quad (1)$$

$$\vec{H}\left(\vec{r}\right) = \vec{H}^{inc}\left(\vec{r}\right) + i\omega\epsilon_0 \int_V \overline{\overline{G}}\left(\vec{r},\vec{r}'\right) \cdot \vec{M}_V\left(\vec{r}'\right) d\vec{r}' + \nabla \times \int_V \overline{\overline{G}}\left(\vec{r},\vec{r}'\right) \cdot \vec{J}_V\left(\vec{r}'\right) d\vec{r}', \ \vec{r} \in V \quad (2)$$

where $\vec{E}^{inc}(\vec{r})$ and $\vec{H}^{inc}(\vec{r})$ are the incident electric field and magnetic field, respectively, while $\vec{E}(\vec{r})$ and $\vec{H}(\vec{r})$ are the total electric field and magnetic field inside the object, respectively. The integral kernel is the dyadic Green's function given by

$$\overline{\overline{G}}\left(\vec{r},\vec{r}'\right) = \left(\overline{\overline{I}} + \frac{\nabla\nabla}{k_0^2}\right) g\left(\vec{r},\vec{r}'\right) \quad (3)$$

where $\overleftrightarrow{I}$ is the identity dyad, k_0 is the free-space wavenumber, and $g(\vec{r},\vec{r}') = e^{ik_0R}/(4\pi R)$ is the scalar Green's function in which $R = |\vec{r}-\vec{r}'|$ is the distance between an observation point $\vec{r}$ and a source point $\vec{r}'$. The unknown functions to be solved are the volumetric electric current density and magnetic current density inside the object, which are related to the total electric field and magnetic field by

$$\vec{J}_V\left(\vec{r}'\right) = i\omega[\epsilon_0 - \epsilon(\vec{r}')]\vec{E}(\vec{r}') \tag{4}$$
$$\vec{M}_V\left(\vec{r}'\right) = i\omega[\mu_0 - \mu(\vec{r}')]\vec{H}(\vec{r}') \tag{5}$$

where $\epsilon(\vec{r}')$ and $\mu(\vec{r}')$ are the permittivity and permeability of the object, respectively. When the conductive objects are treated as dielectric-like objects, the above VIEs are also applicable. If we choose the current densities as the unknown functions to be solved, the above VIEs can be changed into

$$\frac{1}{i\omega[\epsilon_0-\epsilon\left(\vec{r}'\right)]}\vec{J}_V\left(\vec{r}\right)-i\omega\mu_0\int_V\overline{\overline{G}}\left(\vec{r},\vec{r}'\right)\cdot\vec{J}_V\left(\vec{r}'\right)dV'+\int_V\nabla g\left(\vec{r},\vec{r}'\right)\times\vec{M}_V\left(\vec{r}'\right)dV'=\vec{E}^{inc}\left(\vec{r}\right),\ \ \vec{r}\in V \tag{6}$$
$$\frac{1}{i\omega[\mu_0-\mu\left(\vec{r}'\right)]}\vec{M}_V\left(\vec{r}\right)-i\omega\epsilon_0\int_V\overline{\overline{G}}\left(\vec{r},\vec{r}'\right)\cdot\vec{M}_V\left(\vec{r}'\right)dV'-\int_V\nabla g\left(\vec{r},\vec{r}'\right)\times\vec{J}_V\left(\vec{r}'\right)dV'=\vec{H}^{inc}(\vec{r}),\ \ \vec{r}\in V \tag{7}$$

where $\epsilon(\vec{r}') = \epsilon_r'\epsilon_0 + i\frac{\sigma}{\omega} = \epsilon_c$ in which ϵ_r' is the real part of relative permittivity and σ is the conductivity while $\mu(\vec{r}')$ can be assumed to be a constant μ_c for conductive media. The conductive objects with a conductivity σ are usually nonmagnetic or $\mu(\vec{r}') = \mu_0$, the magnetic current density vanishes and the VIEs can be reduced to

$$\frac{1}{i\omega[\epsilon_0-\epsilon(\vec{r}')]}\vec{J}_V(\vec{r}) - i\omega\mu_0\int_V\overline{\overline{G}}(\vec{r},\vec{r}')\cdot\vec{J}_V(\vec{r}')d\vec{r}' = \vec{E}^{inc}(\vec{r}), \quad \vec{r}\in V. \tag{8}$$

3. POINT-MATCHING SCHEME

The above VIE for conductive objects can be solved with a point-matching or collocation scheme. Since the current density is chosen as the unknown function to be solved and the integral kernel, i.e., dyadic Green's function, does not include the material property of the objects, we can greatly facilitate the implementation. The VIEs are first expressed into a scalar form since we do not use any vector basis function. When we assume that the current densities are constant in a small tetrahedral element after discretization, the VIE can be changed into

$$\begin{aligned}&\sum_{n=1}^{N}\int_{\Delta V_n}dV'\left\{i\omega\mu_0\overline{\overline{G}}\left(\vec{r}^c_m,\vec{r}'\right)\cdot\begin{bmatrix}J_V^x\left(\vec{r'}^c_n\right)\\ J_V^y\left(\vec{r'}^c_n\right)\\ J_V^z\left(\vec{r'}^c_n\right)\end{bmatrix}-\begin{bmatrix}g_y\left(\vec{r}^c_m,\vec{r}'\right)M_V^z\left(\vec{r'}^c_n\right)-g_z\left(\vec{r}^c_m,\vec{r}'\right)M_V^y\left(\vec{r'}^c_n\right)\\ g_z\left(\vec{r}^c_m,\vec{r}'\right)M_V^x\left(\vec{r'}^c_n\right)-g_x\left(\vec{r}^c_m,\vec{r}'\right)M_V^z\left(\vec{r'}^c_n\right)\\ g_x\left(\vec{r}^c_m,\vec{r}'\right)M_V^y\left(\vec{r'}^c_n\right)-g_y\left(\vec{r}^c_m,\vec{r}'\right)M_V^x\left(\vec{r'}^c_n\right)\end{bmatrix}\right\}\\&-\frac{1}{i\omega\left(\epsilon_0-\epsilon_c\right)}\begin{bmatrix}J_V^x\left(\vec{r}^c_m\right)\\ J_V^y\left(\vec{r}^c_m\right)\\ J_V^z\left(\vec{r}^c_m\right)\end{bmatrix}=-\begin{bmatrix}E_x^{inc}\left(\vec{r}^c_m\right)\\ E_y^{inc}\left(\vec{r}^c_m\right)\\ E_z^{inc}\left(\vec{r}^c_m\right)\end{bmatrix},\quad m=1,2,\ldots,N\end{aligned} \tag{9}$$

$$\begin{aligned}&\sum_{n=1}^{N}\int_{\Delta V_n}dV'\left\{i\omega\epsilon_0\overline{\overline{G}}\left(\vec{r}^c_m,\vec{r}'\right)\cdot\begin{bmatrix}M_V^x\left(\vec{r'}^c_n\right)\\ M_V^y\left(\vec{r'}^c_n\right)\\ M_V^z\left(\vec{r'}^c_n\right)\end{bmatrix}+\begin{bmatrix}g_y\left(\vec{r}^c_m,\vec{r}'\right)J_V^z\left(\vec{r'}^c_n\right)-g_z\left(\vec{r}^c_m,\vec{r}'\right)J_V^y\left(\vec{r'}^c_n\right)\\ g_z\left(\vec{r}^c_m,\vec{r}'\right)J_V^x\left(\vec{r'}^c_n\right)-g_x\left(\vec{r}^c_m,\vec{r}'\right)J_V^z\left(\vec{r'}^c_n\right)\\ g_x\left(\vec{r}^c_m,\vec{r}'\right)J_V^y\left(\vec{r'}^c_n\right)-g_y\left(\vec{r}^c_m,\vec{r}'\right)J_V^x\left(\vec{r'}^c_n\right)\end{bmatrix}\right\}\\&-\frac{1}{i\omega\left(\mu_0-\mu_c\right)}\begin{bmatrix}M_V^x\left(\vec{r}^c_m\right)\\ M_V^y\left(\vec{r}^c_m\right)\\ M_V^z\left(\vec{r}^c_m\right)\end{bmatrix}=-\begin{bmatrix}H_x^{inc}\left(\vec{r}^c_m\right)\\ H_y^{inc}\left(\vec{r}^c_m\right)\\ H_z^{inc}\left(\vec{r}^c_m\right)\end{bmatrix},\quad m=1,2,\ldots,N\end{aligned} \tag{10}$$

where $\vec{r}^c_m$ represents the center of the mth tetrahedron (observation element) and $\vec{r'}^c_n$ denotes the center of the nth tetrahedron (source element). Also, $\{g_x(\vec{r},\vec{r}'), g_y(\vec{r},\vec{r}'), g_z(\vec{r},\vec{r}')\}$ are the three components of the gradient of scalar Green's function. The matrix elements in the above can be obtained by integrating the components of the dyadic Green's function or the gradient of scalar Green's function over each small tetrahedral element ΔV_n $(n = 1, 2, \ldots, N)$ for different observation point $\vec{r}_m^c$ $(m = 1, 2, \ldots, N)$. The dyadic Green's function has nine components but only six are independent due to its symmetry. If the media are nonmagnetic or $\vec{M}_V(\vec{r}') = 0$, then the above matrix equation can be reduced to

$$i\omega\mu_0 \sum_{n=1}^{N} \left\{ \left[\int_{\Delta V_n} \overline{\overline{G}}(\vec{r}_m^c, \vec{r}') d\vec{r}' \right] \cdot \begin{bmatrix} J_V^x(\vec{r'}_n^c) \\ J_V^y(\vec{r'}_n^c) \\ J_V^z(\vec{r'}_n^c) \end{bmatrix} \right\} - \frac{1}{i\omega(\epsilon_0 - \epsilon_c)} \begin{bmatrix} J_V^x(\vec{r}_m^c) \\ J_V^y(\vec{r}_m^c) \\ J_V^z(\vec{r}_m^c) \end{bmatrix} = - \begin{bmatrix} E_x^{inc}(\vec{r}_m^c) \\ E_y^{inc}(\vec{r}_m^c) \\ E_z^{inc}(\vec{r}_m^c) \end{bmatrix}, \quad m = 1, 2, \ldots, N. \tag{11}$$

When $m \neq n$, the integral kernel is regular and we can use a numerical quadrature rule like the Gauss-Legendre quadrature rule to evaluate the matrix elements. When $m = n$, however, the integral kernel is hypersingular and we need to specially treat it. Since the integrand does not include the material parameters of the objects, we can use the singularity treatment technique developed for regular dielectric media to evaluate the relevant matrix elements [9, 10].

4. NUMERICAL EXAMPLES

We present two numerical examples to demonstrate the proposed approach. The first example considers the EM scattering by a conductive sphere which is embedded in a dielectric medium with a relative permittivity $\epsilon_r = 3.0$. The sphere has a radius $a = 5.0$ mm, a relative permeability $\mu_r = 20.0$, and a conductivity $\sigma = 2.0 \times 10^6$ S/m. The incident wave is excited by a z-polarized magnetic dipole with a unit moment and a frequency $f = 1.0$ KHz. The dipole is located at the $+z$ axis and is 50.0 mm away from the center of the sphere. We calculate the scattered electric field in a near zone, namely, the observation is taken along a line defined by $x = [-10, 10]$ mm, $y = 0$, and $z = 10$ mm. Figure 1 depicts the result when $x = [0, 10]$ mm (the field within $x = [-10, 0]$ mm is symmetric to the one within $x = [0, 10]$ mm) and we can see that it agrees well with the exact Mie-series solution.

The second example illustrates the EM scattering by a conductive sphere coated with a dielectric medium. The conductive sphere has a radius $a_1 = 0.3$ m and a conductivity $\sigma = 1.0$ S/m while the dielectric coating has an outer radius $a_2 = 0.4$ m and the relative permittivity $\epsilon_r = 4.0$. The incident wave is a plane wave propagating along the $+z$ direction and has a frequency $f = 300$ MHz. We calculate the bistatic radar cross section (RCS) observed along the principal cut ($\phi = 0°$ and $\theta = 0°$–$180°$) for the scatterer in both vertical polarization (VV) and horizontal polarization (HH). Figure 2 plots the solutions and it can be seen that they agree with the exact Mie-series solutions

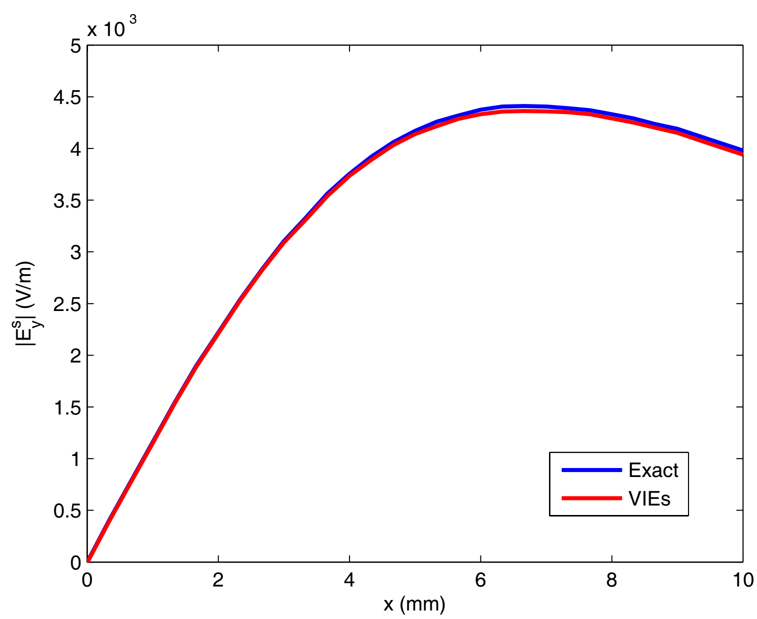

Figure 1: Scattered electric field (y-component) by a conductive sphere with a radius $a = 5.0$ mm and conductivity $\sigma = 2.0 \times 10^6$ S/m. The observation is taken along a line defined by $x = [0, 10.0]$ mm, $y = 0$, and $z = 10.0$ mm.

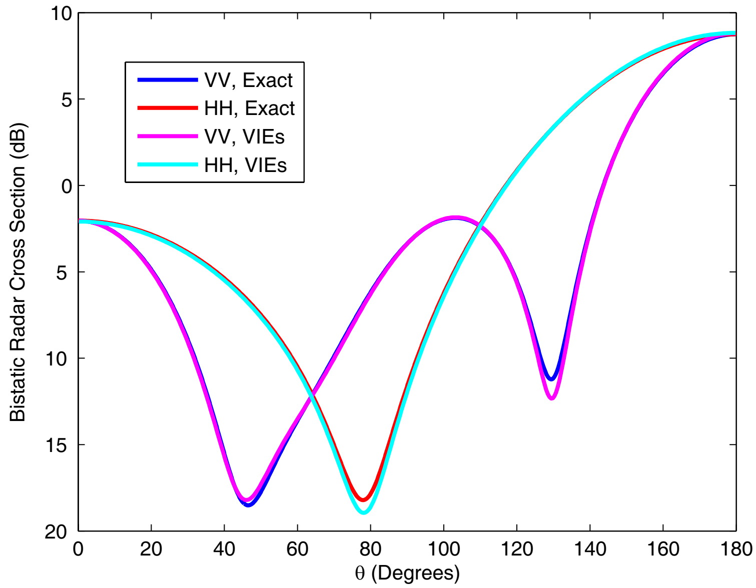

Figure 2: Bistatic RCS solutions for EM scattering by a conductive sphere coated with a dielectric medium. The conductive sphere has a radius $a_1 = 0.3\,\mathrm{m}$ and a conductivity $\sigma = 1.0\,\mathrm{S/m}$ while the dielectric coating has an outer radius $a_2 = 0.4\,\mathrm{m}$ and the relative permittivity $\epsilon_r = 4.0$. The incident wave has a frequency $f = 300\,\mathrm{MHz}$.

very well. Note that the conductivities and frequencies in the two examples are very different, indicating that the approach can accommodate a wide change of conductivity and frequency. Also, the second example includes two very different materials, meaning that the object is very inhomogeneous, but the approach can also use the single VIEs to produce good solutions without a special treatment.

5. CONCLUSION

The EM scattering by conductive objects has been solved by the SIEs with the MoM discretization, but the solutions are susceptible to the choice of frequency and conductivity. In this work, we use the VIEs as governing equations and propose a point-matching scheme to solve the problem. The VIEs are usually well-conditioned and only has one unknown function to be solved for dielectric (nonmagnetic) objects. The point-matching scheme does not rely on any basis and testing functions and can use nonconforming meshes. Furthermore, the scheme can choose the current densities as unknowns and the integral kernels are free of the material parameters of objects so the solutions can be valid in a wide range of frequency and conductivity. A numerical example is presented to demonstrate the approach and its robustness has been seen.

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