Fractal Labyrinths: Path Matrices and Borders Topology

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Abstract— In this paper some elements of the theory of fractal labyrinths are presented using fractional calculus mathematics and random walk model. A model of finitely ramified deterministic labyrinth fractal on a square grid is built, topology of its boundaries is defined, its pathways are calculated using the path matrices up to isomorphism accuracy with respect to rotation and flipping. The path matrices were calculated using the Jordan decomposition. For the constructed model of a fractal labyrinth there have been obtained numerical values of the labyrinth path's lengths before its exits, allowing obtaining the characteristics of the simulated fractal environment.

1. INTRODUCTION

Application of fractal formalism to the analysis of complex weakly formalized processes and systems, problems of stochastic image reconstruction of nanostructures [1], genetic and self-affine synthesis of antenna arrays [2], multidimensional signals, a rich variety of modes of transport etc. undergoes presently intensive development. In this paper we expand the class of fractal-scaling approaches by using self-similar (hierarchical) dendritic structures — fractal labyrinths.

2. ELEMENTS OF THE FRACTAL LABYRINTHS THEORY

Description of fractal structures and processes in fractal media does not fit into the traditional framework of differential equations of integer order. It is considered that the mathematical apparatus of fractal geometry is fractional calculus, which allows transition from ordinary differential equations to fractal equations. At the same time the standard methods of solving differential equations in partial derivatives or calculus of associated moments of transport are also applied to fractional equations. Physically, the operators of integro-differentiation of fractional order $D^{\alpha}[f(t)]$, where $-1 < \alpha < 1$, act as a kind of "filters" that emit only those components that are localized on the fractal (fractional) sets the process under investigation. The presence in the equations of fractional derivative is usually interpreted as a reflection of the special property of the process/system — memory, or non-Markovian (hereditarity).

Anomalous kinetics in fractal structures, including such as the labyrinthine fractals (or fractal labyrinths) — topologically connected dendritic (ramified) structures with fractal dimension $d_f > 1$, whose pathways have scaling, self-similar nature — adequately formulated in terms of a model of the continuous in time random walk, when transport events are subject to wide statistics [3]. Random walks in a fractal environment, i.e., chaotic motion with discontinuous trajectory on the bumps and potholes with abrupt change of direction — characterized by jump length $\lambda(x) = \int\limits_0^\infty dt\psi(x,t)$ and waiting time (between two successive jumps, pauses) $w(t) = \int\limits_{-\infty}^{\infty} dx\psi(x,t)$, where $\lambda(x)dx$ is the probability of jump length in the interval $(x, x+dx)$, and $w(t)dt$ — the probability of the waiting time in the interval $(t,\, t+dt)$. Here the probability density function $\psi(x,t) = \lambda(x)w(t)$, if $\lambda(x)$ and $w(t)$ — independent variables, i.e., random walks are determined by the characteristic waiting time $T = \int\limits_0^\infty dtw(t)t$ and variance of the length of the jump $\sum^2 = \int\limits_{-\infty}^{\infty} dx\lambda(x)x^2$.

Random walks are equivalent to the diffusion. In this connection the transport processes which in the limit of free force are slower than Brownian diffusion, treated as subdiffusion. The probability density function $W(x,t)$ to be in point x at the instant t obeys algebraic relation in the Fourier-Laplace space [4]

$$W(k,u) = \frac{1-w(u)}{u}\frac{W_0(k)}{1-\psi(k,u)}, \tag{1}$$

where $W_0(k)$ — Fourier-transformation of the initial condition $W_0(x)$. In the fractal time of random walk when T diverges (long pause), but the jump — Gaussian (with finite Σ^2), diffusion —

anomalous diffusion is described by the fractional diffusion equation (FDE) [3]:

$$\frac{\partial W}{\partial t} = {}_0D_t^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} W(x,t) \tag{2}$$

where $W(x,t)$ — propagator, which depends in general on the singularities of geometry of the interaction; α — anomalous diffusion index (the order parameter of the fractional derivative, fractal dimension of the environment); K_a — constant of the generalized diffusion (subdiffusion) which is defined as $K_\alpha \equiv \sigma^2/\tau^\alpha$ and in terms of scale σ and τ has the dimension $[K^\alpha] = \mathrm{cm}^2\mathrm{s}^{-\alpha}$; the operator ${}_0D_t^{1-\alpha} = \frac{\partial}{\partial t} {}_0D_t^{-\alpha}$ for $0 < \alpha < 1$ — Riemann-Liouville operator, defined by the integral relation

$${}_0D_t^{1-\alpha} W(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int\limits_0^t dt' \frac{W(x,t')}{(t-t')^{1-\alpha}} \tag{3}$$

— direct continuation of the multiple integral Cauchy for an arbitrary complex α with $\mathrm{Re}(\alpha) > 0$. Thus integrodifferential nature of fractional Riemann-Liouville operator ${}_0D_t^{1-\alpha}$ in accordance with (3), and with the integral kernel of the form $M(t) \propto t^{\alpha-1}$ provides non-Markovian nature of the subdiffusion process which is determined by FDE (2).

Rewriting (2) in the equivalent form

$${}_0D_t^\alpha W - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} W_0(x) = K_\alpha \frac{\partial^2}{\partial x^2} W(x,t), \tag{4}$$

we obtain the initial value $W_0(x)$ in the form of the inverse power law $(t^{-\alpha}/\Gamma(1-\alpha))W_0(x)$, rather than exponential law, as for the standard diffusion [3]. At the same time, in the limit, $\alpha \to 1$ FDE (2) reduces to the second Fick's law, as it should be. The mean square displacement of a random walk is

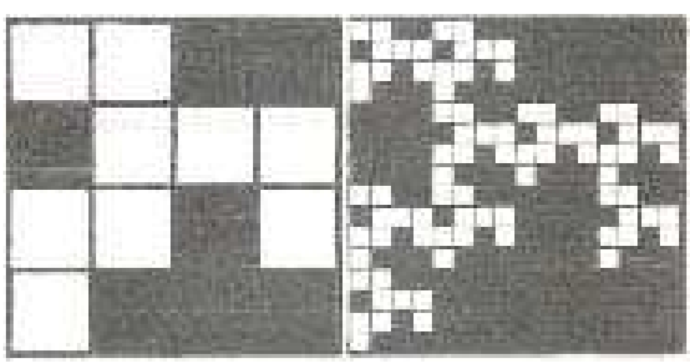
$$\left\langle x^2(t) \right\rangle = \frac{2K_\alpha}{\Gamma(1-\alpha)} t^\alpha. \tag{5}$$

Thus, FDE concept is a real extension of the standard diffusion equation in the same sense in which the random walk model generalizes the Brownian motion.

Anomalous diffusion index α parameterizes several areas of anomalous transport: subdiffusion for $0 < \alpha < 1$, superdiffusion for $\alpha > 1$, the normal Brownian diffusion at the interface between the sub-and superdiffusion with $\alpha = 1$ and a ballistic motion at $\alpha = 2$. Kinetics in fractal labyrinths are based on the description of the phenomenon of subdiffusion, which corresponds to $0 < \alpha < 1$.

3. ALGORITHM CONSTRUCTION OF FRACTAL LABYRINTH

As a generating element of the model fractal let's use the construction of L. L. Christie [5]. Pre-fractal of the first order, $n = 1$ (see Figure 1(a)) is built in the unit square on a 4-node square $m \times m$ grid ($m = 4$) with a characteristic size L. Figure 1(b) — pre-fractal of the 2nd order ($n = 2$), shows the development of the generating element through the recursive branching and scaling.



(a) (b)

Figure 1: 4×4-labyrinth fractals: (a) $n = 1$, (b) $n = 2$.

An iterative procedure for obtaining the fractal set of points $x, y \in [0,1]$ on the unit square $[0,1]\times[0,1]$ for $m \geq 1$ is given by the relations

$$S_{i,j}^{m}=\left\{(x,y)\left|\frac{i}{m}\leq x\leq\frac{i+1}{m},\ \frac{j}{m}\leq y\leq\frac{j+1}{m}\right.\right\}\ \text{and}\ S_m=\left\{S_{i,j}^{m}\mid 0\leq i\leq m-1,\ 0\leq j\leq m-1\right\}.$$

The set of white squares of the first order ($n = 1$) in Figure 1(a) denotes $W_1 \subset S_m$, then the set of black squares of the first order is $B_1 = S_m \backslash W_1$. Herewith

$$\forall (z_x, z_y) \in [0,1]\times[0,1] \exists P_Q(z_x, z_y) = (qz_x + x, qz_y + y).$$

For $n \geq 2$ we have

$$S_{m^n} = \left\{S_{i,j}^{m^n} \mid 0 \leq i \leq m^n - 1, \quad 0 \leq j \leq m^n - 1\right\},$$

wherein respectively

$$W_n = \bigcup\nolimits_{W_1 \in A_1,\, W_{n-1} \in W_{n-1}} \left\{P_{W_{n-1}}(W)\right\}$$

and $B_n = S_{m^n} \backslash W_n$. For $n \geq 1$, we define the associated graph $G(W_n)$ as a self-similar set, the vertices of which are the cells of Figure 1 — white squares, and the edges — the passages between adjacent cells, the sequence of vertices is a path in a graph. If this path is the only path and it is not a cycle, then such connected graph is a tree, a self-similar dendrite. We consider a finitely ramified deterministic fractal on a square grid. Topology of its borders is the following. The top row of order n, of sets W_n on the border of the dendrite is the set of all white squares in $\{S_{i,m^n-1,m^n} | 0 \leq i \leq m^n - 1\}$. The bottom row, left column, and right column are defined by analogy. A top exit in W_n is a white square in the top row of order n, such that there is a white square in the same column in the bottom row of order n. A bottom exit is defined by analogy. A left exit in W_n is a white square in the left column of order n, such that there is a white square in the same row in the right column of order n. A right exit is defined by analogy. Such dendritic set W_n with exits is called *$m \times m$-labyrinth set*. Let us distinguish in W_n ($n \geq 1$) a sequence of compact sets $L_n = \bigcup_{W \in W_n} W$, then we obtain $\{L_n\}_{n=1}^{\infty}$ — monotonously decreasing sequence, whose limit is $L_\infty = \bigcap_{n=1}^{\infty} L_n$. The limit set L_∞ of a labyrinth set W_1 is called *labyrinth fractal*. The Hausdorff dimension of L_∞ is

$$\dim_H(L_\infty) = \frac{\log|W_1|}{\log m},$$

which one obtains from known results for self-similar sets [4]. Since in the case of labyrinth fractals, we have $|W_1| > m$, it follows that $\dim_H > 1$.

In the pre-fractals Figure 1 there are six possible options of paths: the path from vertex to base with the length p_{1n}, which leads to the exit of the W_n; the paths from left to right, top to right, right to bottom, bottom to left and left to top, leading to the exits with lengths p_{2n}, p_{3n}, p_{4n}, p_{5n}, p_{6n}, respectively. Our model of labyrinth is limited, i.e., labyrinth is blocked horizontally and vertically: horizontally — if the series of squares from left to right is contained at least one black square and vertically — if a black square is contained in the column of the squares from top to bottom exit. Based on the existence and uniqueness theorems of curves of finite length in blocked labyrinth fractals [4], let's construct pathways of fractal tree using the path matrix, up to isomorphism accuracy with respect to rotation and flipping. There is a non-negative 6×6 matrix path M, so that for $n \geq 1$

$$\begin{pmatrix} p_{1n} \\ p_{2n} \\ p_{3n} \\ p_{4n} \\ p_{5n} \\ p_{6n} \end{pmatrix} = M \cdot \begin{pmatrix} p_{1n-1} \\ p_{2n-1} \\ p_{3n-1} \\ p_{4n-1} \\ p_{5n-1} \\ p_{6n-1} \end{pmatrix} = M^n \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Figure 1(a) shows the example of 4×4-labyrinth set B that is horizontally and vertically blocked. Let M be the path matrix of the labyrinth set W_1. In the Jordan decomposition $M = B \cdot J \cdot B^{-1}$, we have

$$J = \begin{pmatrix} 000000 \\ 010000 \\ 001000 \\ 000200 \\ 000020 \\ 000005 \end{pmatrix}. \quad \text{Therefore,} \quad M^n = B \cdot J^n \cdot B^{-1}, \quad \text{where} \quad J^n = \begin{pmatrix} 000000 \\ 010000 \\ 001000 \\ 0002^n00 \\ 00002^n0 \\ 000005^n \end{pmatrix}.$$

After algebraic simplifications (explicit forms of B and B^{-1} are omitted) we obtain the numerical values of path lengths (unit is — cell-square) $p_{n1} = (4 \cdot 5^n - 2^n)1/3$, $p_{n2} = (4 \cdot 5^n - 2^n)1/3$, $p_{n3} = 5^n + 2^n - 1$, $p_{n4} = (5^{n+1} - 2^{n+1})1/3$, $p_{n5} = (5^n - 2^n + 3)1/3$, $p_{n6} = 5^n$. Herewith, the matrix J of Jordan decomposition is the same for all horizontal and vertical 4×4-blocked labyrinth sets. For $n = 1$ we have 6, 6, 6, 7, 2, 5; for $n = 2$: 32, 32, 28, 39, 8, 25 — see Figure 1.

Knowling the values of the paths' lengths and of the mean square displacement of a random walk (5) allows us to determine the times of passing through fractal labyrinth, anomalous diffusion index and the fractal dimension of the medium.

4. CONCLUSION

This paper presents the basic relations of the anomalous transport with use of the continuous time random walk scheme as the starting point. These relations determines also the fractal kinetics. To investigate the properties of labyrinthine fractals we constructed the model of fractal labyrinth — dendritic fractal with exits at the boundaries. We formed the paths matrix for such labyrinth considering the topology of its borders. Using the Jordan decomposition, we calculated lengths of all possible paths in the constructed model. The proposed approach can be used to describe transport dynamics in complex systems, which are governed by anomalous diffusion and non-exponential relaxation patterns. Where approximation of processes or structures by dendritic graphs is possible, the calculation of pathways allows receiving fractal characteristics of the investigated medium.

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