Electromagnetic Waves Described with the Complex Quaternion

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Abstract— J. C. Maxwell applied the quaternion analysis and the vector terminology to depict the electromagnetic theory. Nowadays the scholars can use the complex quaternion to study the electromagnetic feature, including the Maxwell's equations and the current continuity equation etc. Meanwhile it is able to describe the gravitational theory with the complex quaternion. On the basis of these studies, the paper separates the complex octonion into two components, the complex quaternion and the complex S-quaternion. The complex quaternion is suitable to describe the gravitational theory, while the complex S-quaternion is proper to depict the electromagnetic theory. Further the paper can apply the complex quaternion to describe the wave equation of gravitational field, and use the complex S-quaternion to depict the wave equation of electromagnetic field. The result reveals that the complex octonion is suitable to describe simultaneously the wave features of electromagnetic and gravitational fields.

1. INTRODUCTION

J. C. Maxwell represented the physical feature of electromagnetic field with the quaternion analysis and the vector terminology. In 1843, W. R. Hamilton invented the quaternion. And J. T. Graves and A. Cayley discovered the octonion independently. Later the scientists and engineers separated the quaternion into the scalar part and vector part. Maxwell mingled naturally the quaternion and the vector to describe the electromagnetic feature in his works. Making use of the complex quaternion in recent years, some scholars begin to study the Maxwell's equations [1] and the current continuity equation [2], and even the physics feature of gravitational field [3].

The ordered couple of quaternions compose the octonion. On the contrary, the octonion is able to be separated into two components, the quaternion and the S-quaternion (short for the second quaternion), while their coordinates are able to be complex numbers. The quaternion space is suitable to depict the gravitational features, and the S-quaternion space is proper to describe the electromagnetic features. Further it can directly deduce the wave equations in the electromagnetic and gravitational fields described with the complex octonion.

In the quaternion space, it is able to infer the field strength and field source etc in the gravitational field, and deduce the gravitational field equations and wave equation etc. The Newton's law of universal gravitation in the classical gravitational theory can be reduced from one of gravitational field equations described with the quaternion. And the gravitational waves are transverse waves in a vacuum. In the S-quaternion space, it is able to deduce the field strength and field source etc in the electromagnetic field. Further the Maxwell's equations and electromagnetic wave equation etc can be inferred directly also. The electromagnetic wave equation is identical with that in the classical electromagnetic theory with the vector terminology.

2. COMPLEX QUATERNION SPACE

The octonion space $\mathbb{E}$ can be separated into two subspaces, the quaternion space $\mathbb{E}_g$ and the S-quaternion space $\mathbb{E}_e$. In the quaternion space $\mathbb{E}_g$ for the gravitational field, the basis vector is $\mathbb{E}_g = (\mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$, the radius vector is $\mathbb{R}_g = ir_0\mathbf{i}_0 + \Sigma r_k\mathbf{i}_k$, and the velocity is $\mathbb{V}_g = iv_0\mathbf{i}_0 + \Sigma v_k\mathbf{i}_k$. The gravitational potential is $\mathbb{A}_g = ia_0\mathbf{i}_0 + \Sigma a_k\mathbf{i}_k$, the gravitational strength is $\mathbb{B}_g = h_0\mathbf{i}_0 + \Sigma h_k\mathbf{i}_k$, and the gravitational source is $\mathbb{S}_g = is_0\mathbf{i}_0 + \Sigma s_k\mathbf{i}_k$. In the S-quaternion space $\mathbb{E}_e$ for the electromagnetic field, the basis vector is $\mathbb{E}_e = (\mathbf{I}_0, \mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3)$, the radius vector is $\mathbb{R}_e = iR_0\mathbf{I}_0 + \Sigma R_k\mathbf{I}_k$, and the velocity is $\mathbb{V}_e = iV_0\mathbf{I}_0 + \Sigma V_k\mathbf{I}_k$. The electromagnetic potential is $\mathbb{A}_e = iA_0\mathbf{I}_0 + \Sigma A_k\mathbf{I}_k$, the electromagnetic strength is $\mathbb{B}_e = H_0\mathbf{I}_0 + \Sigma H_k\mathbf{I}_k$, and the electromagnetic source is $\mathbb{S}_e = iS_0\mathbf{I}_0 + \Sigma S_k\mathbf{I}_k$. Herein $\mathbb{E}_e = \mathbb{E}_g \circ \mathbf{I}_0$. The symbol $\circ$ denotes the octonion multiplication. r_j, v_j, a_j, s_j, R_j, V_j, A_j, S_j, h_0, and H_0 are all real. h_k and H_k are all complex numbers. i is the imaginary unit. $\mathbf{i}_0 = 1$. $\mathbf{i}_k^2 = -1$, $\mathbf{I}_j^2 = -1$. $j = 0,\ 1,\ 2,\ 3$. $k = 1,\ 2,\ 3$.

In the octonion space, $\mathbb{E} = \mathbb{E}_g + \mathbb{E}_e$, the octonion radius vector is $\mathbb{R} = \mathbb{R}_g + k_{eg}\mathbb{R}_e$, the octonion velocity is $\mathbb{V} = \mathbb{V}_g + k_{eg}\mathbb{V}_e$. The octonion field potential is $\mathbb{A} = \mathbb{A}_g + k_{eg}\mathbb{A}_e$, the octonion field strength is $\mathbb{B} = \mathbb{B}_g + k_{eg}\mathbb{B}_e$, with k_{eg} being one coefficient.

The octonion field source $\mathbb{S}$ of the electromagnetic and gravitational fields can be defined as,

$$\mu\mathbb{S} = -(i\mathbb{B}/v_0 + \Box)^* \circ \mathbb{B} = \mu_g \mathbb{S}_g + k_{eg}\mu_e \mathbb{S}_e - (i\mathbb{B}/v_0)^* \circ \mathbb{B}, \tag{1}$$

where μ, μ_g, and μ_e are coefficients. $\mu_g < 0$, and $\mu_e > 0$. $*$ denotes the conjugation of octonion. The quaternion operator is $\Box = i\mathbf{i}_0\partial_0 + \Sigma \mathbf{i}_k \partial_k$. $\nabla = \Sigma \mathbf{i}_k \partial_k$, $\partial_j = \partial/\partial r_j$. $v_0 = \partial r_0/\partial t$. v_0 is the speed of light, and t is the time. $\mathbb{B} = \Box \circ \mathbb{A}$. $\mathbb{B}_g = \Box \circ \mathbb{A}_g$, $\mathbb{B}_e = \Box \circ \mathbb{A}_e$. The gauge conditions of field potential are chosen as $h_0 = 0$ and $H_0 = 0$. In general, the contribution of the tiny term, $(i\mathbb{B}^* \circ \mathbb{B}/v_0)$, could be neglected in the weak fields.

According to the coefficient k_{eg} and the basis vectors, $\mathbb{E}_g$ and $\mathbb{E}_e$, the octonion definition of the field source can be separated into two parts,

$$\mu_g \mathbb{S}_g = -\Box^* \circ \mathbb{B}_g, \tag{2}$$

$$\mu_e \mathbb{S}_e = -\Box^* \circ \mathbb{B}_e, \tag{3}$$

where a comparison with the classical electromagnetic and gravitational theories reveal that, $\mathbb{S}_g = m\mathbb{V}_g$, and $\mathbb{S}_e = q\mathbb{V}_e$, in the case for single one particle. m is the mass density, while q is the density of electric charge.

3. FIELD EQUATIONS

In the quaternion space $\mathbb{E}_g$, the definition of gravitational source, Eq. (2), can be expressed as,

$$-\mu_g(is_0 + \mathbf{s}) = (i\partial_0 + \nabla)^* \circ (i\mathbf{g}/v_0 + \mathbf{b}), \tag{4}$$

where $\mu_g = 1/(\varepsilon_g v_0^2)$, and $\varepsilon_g = -1/(4\pi G)$, with G being the gravitational constant. $\mathbf{a} = \Sigma a_k \mathbf{i}_k$, $a_0 = \varphi/v_0$, with φ being the scalar potential of gravitational field. $\mathbb{B}_g = \Sigma h_k \mathbf{i}_k = i\mathbf{g}/v_0 + \mathbf{b}$, when $h_0 = 0$. The gravitational acceleration is, $\mathbf{g}/v_0 = \partial_0 \mathbf{a} + \nabla a_0$. $\mathbf{b} = \nabla \times \mathbf{a}$. $\mathbf{g} = \Sigma g_k \mathbf{i}_k$, $\mathbf{b} = \Sigma b_k \mathbf{i}_k$. $\mathbf{s} = \Sigma s_k \mathbf{i}_k$.

The above can be expanded into the following equations,

$$\nabla^* \cdot \mathbf{b} = 0, \tag{5}$$

$$\partial_0 \mathbf{b} + \nabla^* \times \mathbf{g}/v_0 = 0, \tag{6}$$

$$\nabla^* \cdot \mathbf{g}/v_0 = -\mu_g s_0, \tag{7}$$

$$\nabla^* \times \mathbf{b} - \partial_0 \mathbf{g}/v_0 = -\mu_g \mathbf{s}. \tag{8}$$

Therefore Eqs. (5)–(8) are the gravitational field equations. Because the gravitational constant G is weak and the velocity ratio $\mathbf{v}/v_0$ is tiny, the gravity produced by the linear momentum $\mathbf{s}$ can be ignored in general. When $\mathbf{b} = 0$ and $\mathbf{a} = 0$, Eq. (7) can be degenerated into the Newton's law of universal gravitation in the classical gravitational theory.

In the S-quaternion space $\mathbb{E}_e$, the definition of electromagnetic source, Eq. (3), is written as,

$$-\mu_e(i\mathbf{S}_0 + \mathbf{S}) = (i\partial_0 + \nabla)^* \circ (i\mathbf{E}/v_0 + \mathbf{B}), \tag{9}$$

where the electromagnetic coefficient is $\mu_e = 1/(\varepsilon_e v_0^2)$, with ε_e being the permittivity. $\mathbf{A} = \Sigma A_k \mathbf{I}_k$. $\mathbf{A}_0 = A_0 \mathbf{I}_0$, $A_0 = \phi/v_0$, with ϕ being the scalar potential of electromagnetic field. $\mathbb{B}_e = i\mathbf{E}/v_0 + \mathbf{B}$, when $H_0 = 0$. The electric field intensity is, $\mathbf{E}/v_0 = \partial_0 \mathbf{A} + \nabla \circ \mathbf{A}_0$, and the magnetic flux density is, $\mathbf{B} = \nabla \times \mathbf{A}$. $\mathbf{E} = \Sigma E_k \mathbf{I}_k$, $\mathbf{B} = \Sigma B_k \mathbf{I}_k$. $\mathbf{S}_0 = S_0 \mathbf{I}_0$. $\mathbf{S} = \Sigma S_k \mathbf{I}_k$.

The above can be expanded into the following electromagnetic field equations,

$$\nabla^* \cdot \mathbf{B} = 0, \tag{10}$$

$$\partial_0 \mathbf{B} + \nabla^* \times \mathbf{E}/v_0 = 0, \tag{11}$$

$$\nabla^* \cdot \mathbf{E}/v_0 = -\mu_e \mathbf{S}_0, \tag{12}$$

$$\nabla^* \times \mathbf{B} - \partial_0 \mathbf{E}/v_0 = -\mu_e \mathbf{S}, \tag{13}$$

where there may be, $\mathbb{V}_e = \mathbb{V}_g \circ \mathbf{I}(\mathbf{I}_j)$, for the charged particles. And the unit $\mathbf{I}(\mathbf{I}_j)$ is one function of $\mathbf{I}_j$, with $\mathbf{I}(\mathbf{I}_j)^* \circ \mathbf{I}(\mathbf{I}_j) = 1$.

On the analogy of the coordinate definition of complex coordinate system, one can define the coordinate of octonion, which involves the quaternion and S-quaternion simultaneously. In the octonion coordinate system, the octonion physics quantity can be defined as $\{(ic_0 + id_0\mathbf{I}_0) \circ \mathbf{i}_0 + \Sigma(c_k + d_k\mathbf{I}_0^*) \circ \mathbf{i}_k\}$, It means that there are the quaternion coordinate c_k and the S-quaternion coordinate $d_k\mathbf{I}_0^*$ for the basis vector $\mathbf{i}_k$, while the quaternion coordinate c_0 and the S-quaternion coordinate $d_0\mathbf{I}_0$ for the basis vector $\mathbf{i}_0$. Herein c_i and d_i are all real.

Table 1: The multiplication of the operator and octonion physics quantity.

definition	expression meaning
$\nabla \cdot \mathbf{a}$	$-(\partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3)$
$\nabla \times \mathbf{a}$	$\mathbf{i}_1(\partial_2 a_3 - \partial_3 a_2) + \mathbf{i}_2(\partial_3 a_1 - \partial_1 a_3) + \mathbf{i}_3(\partial_1 a_2 - \partial_2 a_1)$
∇a_0	$\mathbf{i}_1 \partial_1 a_0 + \mathbf{i}_2 \partial_2 a_0 + \mathbf{i}_3 \partial_3 a_0$
$\partial_0 \mathbf{a}$	$\mathbf{i}_1 \partial_0 a_1 + \mathbf{i}_2 \partial_0 a_2 + \mathbf{i}_3 \partial_0 a_3$
$\nabla \cdot \mathbf{A}$	$-(\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3)\mathbf{I}_0$
$\nabla \times \mathbf{A}$	$-\mathbf{I}_1(\partial_2 A_3 - \partial_3 A_2) - \mathbf{I}_2(\partial_3 A_1 - \partial_1 A_3) - \mathbf{I}_3(\partial_1 A_2 - \partial_2 A_1)$
$\nabla \circ \mathbf{A}_0$	$\mathbf{I}_1 \partial_1 A_0 + \mathbf{I}_2 \partial_2 A_0 + \mathbf{I}_3 \partial_3 A_0$
$\partial_0 \mathbf{A}$	$\mathbf{I}_1 \partial_0 A_1 + \mathbf{I}_2 \partial_0 A_2 + \mathbf{I}_3 \partial_0 A_3$

4. WAVE EQUATION

4.1. Gravitational Wave Equation

In the quaternion space $\mathbb{E}_g$, for the gravitational wave with the angular frequency ω, the gravitational strength should be a harmonic function with respect to the time, and can be chosen as the function, $\cos(\omega t)$. In order to operate handily and cover more physics contents, it is convenient to adopt the complex vector to represent the gravitational strength,

$$\mathbf{b}(\mathbf{r}, \omega t) = \mathbf{b}(\mathbf{r}) \exp(-i\omega t), \quad \mathbf{g}(\mathbf{r}, \omega t) = \mathbf{g}(\mathbf{r}) \exp(-i\omega t), \tag{14}$$

where $\mathbf{r} = \Sigma r_k \mathbf{i}_k$.

For one angular frequency ω, substituting the above in the gravitational field equations in the vacuum. Deleting the time factor, $\exp(-i\omega t)$, due to the uniform transmission medium, one obtains the following equations,

$$\nabla \cdot \mathbf{b}(\mathbf{r}) = 0, \tag{15}$$
$$(-i\omega)\mathbf{b}(\mathbf{r}) + \nabla^* \times \mathbf{g}(\mathbf{r}) = 0, \tag{16}$$
$$\nabla \cdot \mathbf{g}(\mathbf{r}) = 0, \tag{17}$$
$$\nabla^* \times \mathbf{b}(\mathbf{r}) + (i\omega)\mathbf{g}(\mathbf{r})/v_0^2 = 0. \tag{18}$$

Combining with the above, the cross product of the operator ∇ with Eq. (16) yields,

$$(\omega/v_0)^2 \mathbf{g}(\mathbf{r}) - \nabla^2 \mathbf{g}(\mathbf{r}) = 0, \tag{19}$$

where $\nabla \times (\nabla \times \mathbf{g}(\mathbf{r})) = -\nabla(\nabla \cdot \mathbf{g}(\mathbf{r})) + \nabla^2 \mathbf{g}(\mathbf{r})$, and $\nabla^2 \mathbf{g}(\mathbf{r}) = -\Sigma \partial_k^2 \mathbf{g}(\mathbf{r})$.

Similarly the output of the curl operation of Eq. (18) produces,

$$(\omega/v_0)^2 \mathbf{b}(\mathbf{r}) - \nabla^2 \mathbf{b}(\mathbf{r}) = 0. \tag{20}$$

Equations (19) and (20) constitute the Helmholtz equations for the gravitational wave. Obviously their solutions are,

$$\mathbf{b}(\mathbf{r}) = \mathbf{b}_0 \exp(-i\mathbf{k} \cdot \mathbf{r}), \quad \mathbf{g}(\mathbf{r}) = \mathbf{g}_0 \exp(-i\mathbf{k} \cdot \mathbf{r}), \tag{21}$$

where $\mathbf{b}_0$ and $\mathbf{g}_0$ both are the constant vectors, which are independent to the coordinate and the time. The wave vector is, $\mathbf{k} = \Sigma k_k \mathbf{i}_k$.

Therefore the two parts, $\mathbf{g}(\mathbf{r}, \omega t)$ and $\mathbf{b}(\mathbf{r}, \omega t)$, of the gravitational wave can be written as,

$$\mathbf{b}(\mathbf{r}, \omega t) = \mathbf{b}_0 \exp(-i\mathbf{k} \cdot \mathbf{r} - i\omega t), \quad \mathbf{g}(\mathbf{r}, \omega t) = \mathbf{g}_0 \exp(-i\mathbf{k} \cdot \mathbf{r} - i\omega t), \tag{22}$$

where only the scalar part of gravitational wave is measurable in the experiment.

Further Eq. (15) and Eq. (17) deduce,

$$\mathbf{k} \cdot \mathbf{b}_0 = 0, \quad \mathbf{k} \cdot \mathbf{g}_0 = 0, \tag{23}$$

meanwhile Eq. (16) and Eq. (18) yield,

$$\mathbf{b}_0 = (1/\omega)\mathbf{k} \times \mathbf{g}_0, \quad \mathbf{g}_0 = -(v_0^2/\omega)\mathbf{k} \times \mathbf{b}_0. \tag{24}$$

The above means that the vibration direction of the gravitational wave is perpendicular to the direction of wave propagation. And the gravitational wave is the transverse wave in a vacuum. Meanwhile these three vectors, $\mathbf{b}_0$, $\mathbf{g}_0$, and $\mathbf{k}$, are perpendicular to each other, and obey the Right-handed screw rule.

On the analogy of the classical electromagnetic theory, the paper will introduce the concept of 'refractive index-like' among different transmission media for the gravitational waves. Making use of the concept of 'refractive index-like', it is able to deduce further some wave features from the above, including the 'Reflection law', 'Refraction law', 'Fresnel law', and 'Total reflection' etc for the gravitational waves.

4.2. Electromagnetic Wave Equation

In the S-quaternion space $\mathbb{E}_e$, for the electromagnetic wave with the angular frequency ω, the electromagnetic strength should be a harmonic function with respect to the time, and can be chosen as the function, $\cos(\omega t)$. In order to operate handily and cover more physics contents, it is convenient to adopt the complex vector to represent the electromagnetic strength,

$$\mathbf{B}(\mathbf{r},\omega t)=\mathbf{B}(\mathbf{r})\exp(-i\omega t),\quad \mathbf{E}(\mathbf{r},\omega t)=\mathbf{E}(\mathbf{r})\exp(-i\omega t). \tag{25}$$

For one angular frequency ω, substituting the above in the electromagnetic field equations in the vacuum. Deleting the time factor, $\exp(-i\omega t)$, due to the uniform transmission medium, one obtains the following equations,

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = 0, \tag{26}$$
$$(-i\omega)\mathbf{B}(\mathbf{r})+\nabla^*\times\mathbf{E}(\mathbf{r}) = 0, \tag{27}$$
$$\nabla\cdot\mathbf{E}(\mathbf{r}) = 0, \tag{28}$$
$$\nabla^*\times\mathbf{B}(\mathbf{r})+(i\omega)\mathbf{E}(\mathbf{r})/v_0^2 = 0. \tag{29}$$

Combining with the above, the cross product of the operator ∇ with Eq. (27) yields,

$$(\omega/v_0)^2\mathbf{E}(\mathbf{r})-\nabla^2\mathbf{E}(\mathbf{r})=0, \tag{30}$$

where $\nabla\times(\nabla\times\mathbf{E}(\mathbf{r}))=-\nabla(\nabla\cdot\mathbf{E}(\mathbf{r}))+\nabla^2\mathbf{E}(\mathbf{r})$, and $\nabla^2\mathbf{E}(\mathbf{r})=-\Sigma\partial_k^2\mathbf{E}(\mathbf{r})$.

Similarly the output of the curl operation of Eq. (29) produces,

$$(\omega/v_0)^2\mathbf{B}(\mathbf{r})-\nabla^2\mathbf{B}(\mathbf{r})=0. \tag{31}$$

Equations (30) and (31) constitute the Helmholtz equations for the electromagnetic wave. Obviously their solutions are,

$$\mathbf{B}(\mathbf{r})=\mathbf{B}_0\exp(-i\mathbf{k}\cdot\mathbf{r}),\quad \mathbf{E}(\mathbf{r})=\mathbf{E}_0\exp(-i\mathbf{k}\cdot\mathbf{r}), \tag{32}$$

where $\mathbf{B}_0$ and $\mathbf{E}_0$ both are the constant vectors.

Therefore the magnetic field component $\mathbf{B}(\mathbf{r},\omega t)$ and the electric field component $\mathbf{E}(\mathbf{r},\omega t)$ of the electromagnetic wave can be written as,

$$\mathbf{B}(\mathbf{r},\omega t)=\mathbf{B}_0\exp(-i\mathbf{k}\cdot\mathbf{r}-i\omega t),\quad \mathbf{E}(\mathbf{r},\omega t)=\mathbf{E}_0\exp(-i\mathbf{k}\cdot\mathbf{r}-i\omega t), \tag{33}$$

where only the scalar part of electromagnetic wave is measurable in the experiment.

Further Eq. (26) and Eq. (28) deduce,

$$\mathbf{k}\cdot\mathbf{B}_0=0,\quad \mathbf{k}\cdot\mathbf{E}_0=0, \tag{34}$$

meanwhile Eq. (27) and Eq. (29) yield,

$$\mathbf{B}_0=(1/\omega)\mathbf{k}\times\mathbf{E}_0,\quad \mathbf{E}_0=-(v_0^2/\omega)\mathbf{k}\times\mathbf{B}_0. \tag{35}$$

The above means that the vibration direction of the electromagnetic wave is perpendicular to the direction of wave propagation. And the electromagnetic wave is the transverse wave in a vacuum. Meanwhile these three vectors, $\mathbf{B}_0$, $\mathbf{E}_0$, and $\mathbf{k}$, are perpendicular to each other, and obey the Right-handed screw rule.

Similar to the classical electromagnetic theory, by means of the concept of refractive index, it is able to deduce further some wave features from the above, including the Reflection law, Refraction law, Fresnel law, and Total reflection etc for the electromagnetic waves.

5. CONCLUSIONS

In the complex quaternion space, it is able to describe the field equations and wave equations in the gravitational field. Making use of the concept of 'refractive index-like', it is capable of inferring the 'Reflection law', 'Refraction law', 'Fresnel law', and 'Total reflection' etc among different transmission media for the gravitational waves. In the vacuum, the gravitational wave is the transverse wave. The wave vector is perpendicular to each one of two orthogonal components of gravitational wave. And they satisfy the Right-handed screw rule.

In the complex S-quaternion space, the paper can represent the Maxwell's equations and Helmholtz equations in the electromagnetic field. These equations are respectively identical with that in the classical electromagnetic theory described with the vector terminology. One can conclude the above inferences, expressing corresponding equations into the scalar equations, and then comparing them. By means of the concept of refractive index, it is able to deduce the Reflection law, Refraction law, Fresnel law, and Total reflection etc among different transmission media for the electromagnetic waves.

It should be noted that the paper discussed only some simple features of electromagnetic and gravitational waves described with the complex octonion. But it is clear to show that the application of the complex quaternion can describe the wave feature of gravitational field, while the introduction of the complex S-quaternion can depict the wave feature of electromagnetic field. And these researches set the foundation for the further study about the wave feature. In the following study, it is going to apply the complex octonion to represent the propagation characteristics of electromagnetic and gravitational waves inside the transmission medium.

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