Properties of the Gram Matrices Associated with Loop-flower Basis Functions

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Abstract— In this paper, loop-flower basis functions are addressed to solve the electric field integral equation (EFIE) in electromagnetic scattering issues that rise from perfectly conducting objects. A loop basis function and a flower basis function are both defined as the sum of a set of modified RWG basis functions associated with a node in triangular meshes. Moreover, they could also be represented by node based Lagrange interpolation polynomials. A flower basis function, which also resembles the star basis function, is named after its shape. In contrast to all previous quasi-Helmholtz decomposition, loop-flower decomposition holds several good characteristics. First, loop-flower decomposition can be used to cure low-frequency breakdown of EFIE spectrum. Second, it can also be directly used to implement Calderón preconditioners for EFIE. Last but not least, the unknown number corresponding with loop-flower basis functions reduces to approximately two thirds of the one associated with RWG. Given that the property of a Calderón preconditioner is largely affected by the Gram matrix that links the range and domain of EFIE operator, this paper will focus on the properties of Gram matrices associated with loop-flower basis function. Analysis shows that the Gram matrices associated with loop-flower basis functions are invertible, and their condition numbers are approximately of the order of (h^{-2}), in which h is the characteristic dimension of a triangular mesh. The theoretical analysis will be demonstrated by several numerical examples.

1. INTRODUCTION

Surface integral equations (SIEs) have been widely used for solving electromagnetic radiation and scattering problems from perfect electric conductors (PEC). To numerically solve SIEs with method of moment (MoM) requires surface to be discretized as simple elements such as triangles and quadrilaterals. Rao-Wilton-Glisson (RWG) basis functions are popular to expand surface electric and magnetic currents among divergence-conforming vector basis functions. However, the use of RWG basis functions for the electric field integral equation (EFIE) exhibits low frequency breakdown, that is to say, at very low frequencies, the EFIE discretized system matrix is highly ill-conditioned and hence is difficult to be solved accurately and efficiently. The quasi Helmholtz decomposition, expanding current with loop-tree or loop-star basis functions [1–3], has been proposed to overcome the low frequency breakdown. The use of loop-tree and loop-star basis functions can provide an effective approach to scale the solenoidal currents and the non-solenoidal ones properly at low frequencies, hence, both the two parts of the currents can be handled correctly. However, the quasi Helmholtz decomposition does not improve the spectrum property of the electric field integral operator. Recently, a Calderón multiplicative preconditioner [4] has been proposed to solve the above problem, which largely improves the characteristics of the spectrum of EFIF operator. The main difficulty in implementing the Calderón preconditioner is to find a well conditioned Gram matrix which links the domain and range of the EFIE operators. The Buffa-Christiansen basis functions are approximately orthogonal to the standard RWG basis functions [5], consequently, the resultant Gram matrix of these two basis functions is well conditioned. However, it is necessary to divide the combined operator into four parts and discarding the most hypersingular term, otherwise, the Calderón preconditioner with BC basis functions may fail at very low frequency. Loop-flower basis functions [6] are proposed to implement Calderón preconditioner directly, which work well to overcome the low frequency breakdown. Loop and flower basis functions are both defined on nodes, which lower the degree of freedom compared with RWG basis functions. This work will analyze the condition number of the Gram matrices associated with loop-flower basis functions. Theoretic analysis and numerical results show that the conditioning of loop-flower Gram matrices will become worse with the decreasing of mesh length. In addition, this work will also present numerical results to demonstrate the practical effectiveness of loop-flower basis functions in solving electromagnetic scattering issues.

2. LOOP-FLOWER BASIS FUNCTIONS

Loop-flower basis functions arise from the need to perform quasi Helmholtz decomposition. Loop basis functions are used to expand the solenoidal current, whereas flower basis functions are for the irrotational part. A loop basis function is defined on each interior vertex v_n, which is shown in Fig. 1(a). Explicit representation of loop basis function is as follows

$$\mathbf{\Lambda}_j(\mathbf{r}) = \sum_{i=1}^{N_j} \bar{\bar{\mathbf{\Lambda}}}_{i,j}\mathbf{f}'_j \tag{1}$$

where N_j is the number of the surrounding triangles, $\bar{\bar{\mathbf{\Lambda}}}$ is the loop to RWG transformation matrix, $\mathbf{f}'_i$ is the standard RWG basis function divided by the common edge length.

It is very interesting to find that the loop basis function that we use in the MoM community maintains a direct relationship with the piecewise linear Lagrange basis function [3] (or the nodal basis function) associated with vertex in the FEM community. Consequently, loop basis function can be also expressed by

$$\mathbf{\Lambda}_p(\mathbf{r}) = \nabla \times \hat{\mathbf{n}}_r \lambda_p(\mathbf{r}) = \hat{\mathbf{n}}_r \times \nabla\lambda_p(\mathbf{r}) \tag{2}$$

in which $\hat{\mathbf{n}}_r$ denotes the outward normal unit vector and $\lambda(r)$ denotes piecewise linear Lagrange basis function.

A flower basis function is also defined on the node v_n, like a loop basis function, as depicted in Fig. 1(b). Its support covers all the RWG bases that share the reference node v_n as one free vertex. It can be explicit expressed by

$$\mathbf{f}_j^{\text{flower}}(\mathbf{r}) = \sum_{i=1}^{N_j} \bar{\bar{\boldsymbol{F}}}_{i,j}\mathbf{f}'_i \tag{3}$$

in which, $\bar{\bar{\boldsymbol{F}}}$ stands for the flower to RWG transformation matrix. The reference direction of flower basis function points away from the reference node.

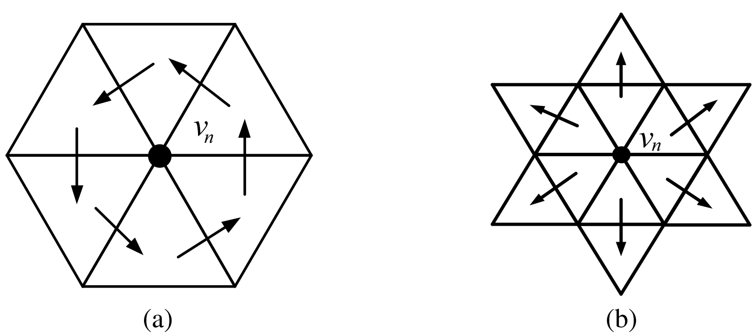

Figure 1: (a) Loop and (b) flower basis function.

3. LOOP-FLOWER GRAM MATRIX

The Gram matrix of RWG bases

$$\left(\bar{\bar{\mathbf{G}}}\right)_{i,j} = \langle \mathbf{f}'_i, \mathbf{f}'_j \rangle \tag{4}$$

is well-conditioned [1], namely, $\bar{\bar{\mathbf{G}}} \asymp \bar{\bar{\mathbf{I}}}$, where $\bar{\bar{\mathbf{I}}}$ is the identity matrix.

When the loop-flower bases are used, the loop-flower to RWG transformation matrix can be written as $\bar{\bar{\mathbf{H}}}^{\Lambda F} = [\bar{\bar{\mathbf{\Lambda}}}, \bar{\bar{\boldsymbol{F}}}]$, whose columns are the coefficients of the loop-flower functions when expressed as linear combinations of RWG functions. Since coefficients of loop and flower functions are orthogonal and RWG Gram matrix is well conditioned, thereby loop-flower Gram matrix meets the following relations

$$\bar{\bar{\mathbf{G}}}^{\Lambda F} = \bar{\bar{\mathbf{H}}}_{\Lambda F}^H \bar{\bar{\mathbf{G}}} \bar{\bar{\mathbf{H}}}_{\Lambda F} \asymp \bar{\bar{\mathbf{H}}}_{\Lambda F}^H \bar{\bar{\mathbf{H}}}_{\Lambda F} = \begin{pmatrix} \bar{\bar{\mathbf{\Lambda}}}^H \bar{\bar{\mathbf{\Lambda}}} & \bar{\bar{\mathbf{0}}} \\ \bar{\bar{\mathbf{0}}} & \bar{\bar{\mathbf{\Sigma}}}^H \bar{\bar{\mathbf{\Sigma}}} \end{pmatrix} \asymp \begin{pmatrix} \bar{\bar{\mathbf{\Lambda}}}^H \bar{\bar{\mathbf{G}}} \bar{\bar{\mathbf{\Lambda}}} & \bar{\bar{\mathbf{0}}} \\ \bar{\bar{\mathbf{0}}} & \bar{\bar{\boldsymbol{F}}}^H \bar{\bar{\mathbf{G}}} \bar{\bar{\boldsymbol{F}}} \end{pmatrix} = \begin{pmatrix} \bar{\bar{\mathbf{G}}}^{\Lambda} & \bar{\bar{\mathbf{0}}} \\ \bar{\bar{\mathbf{0}}} & \bar{\bar{\mathbf{G}}}^{F} \end{pmatrix} \tag{5}$$

where

$$\left\{\bar{\bar{\mathbf{G}}}^{\Lambda}\right\}_{i,j}=\left\{\bar{\bar{\boldsymbol{\Lambda}}}^{H}\bar{\bar{\mathbf{G}}}\bar{\bar{\boldsymbol{\Lambda}}}\right\}_{i,j}=\left\langle\bar{\bar{\boldsymbol{\Lambda}}}_{i},\bar{\bar{\boldsymbol{\Lambda}}}_{j}\right\rangle \quad \text{and} \quad \left\{\bar{\bar{\mathbf{G}}}^{F}\right\}_{i,j}=\left\{\bar{\bar{\boldsymbol{F}}}^{H}\bar{\bar{\mathbf{G}}}\bar{\bar{\boldsymbol{F}}}\right\}_{i,j}=\langle\bar{\bar{\boldsymbol{F}}}_{i},\bar{\bar{\boldsymbol{F}}}_{j}\rangle \tag{6}$$

From (5), it is clear to find that, instead of analyzing the conditioning of the block diagonal loop-flower Gram matrix directly, we can analyze the loop and Gram matrices separately to get the total conditioning.

Substitute (2) into (6), the loop Gram matrix [7] can be rewritten as

$$\left\{\bar{\bar{\mathbf{G}}}^{\Lambda}\right\}_{i,j}=\langle\nabla\times\hat{\mathbf{n}}_r\lambda_i,\,\nabla\times\hat{\mathbf{n}}_r\lambda_j\rangle=\langle\nabla_s\lambda_i,\,\nabla_s\lambda_j\rangle \tag{7}$$

Consider that in a Sobolev space $H^1(\Omega)$, which is the space of functions defined on the Ω that are square integrable and whose gradient is also square integrable Poincare Inequality and Inverse Inequality for piecewise polynomials hold true [8]. Since piecewise linear Lagrange basis function λ meets above requirements, so we have

$$\|\lambda\|_2^2\lesssim\|\nabla\lambda\|_2^2 \quad \text{and} \quad \|\nabla\lambda\|_2^2\lesssim\frac{1}{h^2}\|\lambda\|_2^2 \quad \text{with} \quad \forall\lambda:\lambda=\sum_i\upsilon_i\lambda_i \tag{8}$$

Assume that $\lambda=\sum\limits_i\upsilon_i\lambda_i$, $\|\lambda\|_2^2$ and $\|\nabla\lambda\|_2^2$ can be further expressed as

$$\|\lambda\|_2^2=\bar{\mathbf{v}}^H\bar{\bar{\mathbf{G}}}^{\lambda}\bar{\mathbf{v}} \quad \text{and} \quad \|\nabla\lambda\|_2^2=\bar{\mathbf{v}}^H\bar{\bar{\mathbf{G}}}\bar{\mathbf{v}} \tag{9}$$

Since the Rayleigh Quotient of a matrix is bounded by the maximum and minimum signular values, combing (7), (8) and (9) yields

$$\bar{\bar{\mathbf{G}}}^{\lambda}\lesssim\bar{\bar{\mathbf{G}}}^{\Lambda}\lesssim\frac{1}{h^2}\bar{\bar{\mathbf{G}}}^{\lambda} \tag{10}$$

With the help of $\bar{\bar{\mathbf{G}}}^{\lambda}\asymp h^2\bar{\bar{\mathbf{I}}}$ (readers can refer to [8] and references therein for detail discussion), we can get

$$h^2\bar{\bar{\mathbf{I}}}\lesssim\bar{\bar{\mathbf{G}}}^{\Lambda}\lesssim\bar{\bar{\mathbf{I}}} \tag{11}$$

In result, the condition number of loop Gram matrix satisfies

$$\text{cond}\left(\bar{\bar{\mathbf{G}}}^{\Lambda}\right)\asymp\frac{1}{h^2} \tag{12}$$

When analyze the flower Gram matrix, we will the use the BC basis function as auxiliary tool. In fact, the BC Gran matrix $(\bar{\bar{\mathbf{G}}}_{\rm BC})_{i,j}=\langle\mathbf{f}_i^{\rm BC},\,\mathbf{f}_j^{\rm BC}\rangle$ is well conditioned [5], so that we can get

$$\left(\bar{\bar{\mathbf{G}}}^{F}\right)=\bar{\bar{F}}^{T}\bar{\bar{\mathbf{G}}}\bar{\bar{F}}\asymp\bar{\bar{F}}^{T}\bar{\bar{\mathbf{G}}}_{\rm BC}\bar{\bar{F}}=\left\langle\nabla_s\lambda^{\rm bar},\,\nabla_s\lambda^{\rm bar}\right\rangle \tag{13}$$

where $\lambda^{\rm bar}$ is the piecewise linear, cell-centered function similar to the linear Lagrange basis function. Replacing λ with $\lambda^{\rm bar}$ in the analysis of loop Gram matrix yields

$$\text{cond}\left(\bar{\bar{\mathbf{G}}}^{F}\right)\asymp\frac{1}{h^2} \tag{14}$$

From Equations (5), (12) and (14), it is clear to find that the loop-flower Gram matrix have a condition number which will be growing with mesh refinement as $1/h^2$.

4. APPLICATION OF LOOP-FLOWER BASIS FUNCTIONS

Consider a electromagnetic wave scattering problem by an PEC object with surface Ω whose normal is denoted as $\hat{\mathbf{n}}$. The relationship between incident electric field $\mathbf{E}^{inc}$ and scattered filed $\mathbf{E}^{sca}$, which is produced by an equivalent current $\mathbf{J}(r)$ on the Ω, can be expressed as

$$\hat{\mathbf{n}}\times\mathbf{E}^{sca}=\mathcal{T}(\mathbf{J})=-\hat{\mathbf{n}}\times\mathbf{E}^{inc} \tag{15}$$

in which, the EFIE operator $\mathcal{T}(\mathbf{J}) = \mathcal{T}_s(\mathbf{J}) + \mathcal{T}_h(\mathbf{J})$ with

$$\mathcal{T}_s(\mathbf{J}) = iw\mu\hat{\mathbf{n}}_r \times \int_S \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\mathbf{J}(\mathbf{r}')ds' \tag{16}$$

$$\mathcal{T}_h(\mathbf{J}) = -\frac{1}{i\omega\varepsilon}\hat{\mathbf{n}}_r \times \nabla \int_S \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\nabla'_S \cdot \mathbf{J}(\mathbf{r}')ds' \tag{17}$$

where k stands for the free-space wave number, the subscript "s" and "h" stand for "singular" (vector potential) and "hyper-singular" (scalar potential), respectively. Current density $\mathbf{J}$ is expanded with loop-flower bases as

$$\mathbf{J} = \sum_{n=1}^{N_{\text{loop}}} I_n^l \mathbf{f}_n^{\text{loop}} + \sum_{n=1}^{N_{\text{flower}}} I_n^f \mathbf{f}_n^{\text{flower}} \tag{18}$$

The degree of freedom of loop and flower basis functions can be determined according to [6] and reference therein. Given that the ill-conditioning of impedance matrix is rooted in the spectral property of the EFIE operator, Calderon identity is used to modify the spectrum distribution of EFIE operator. The Calderon identity

$$\mathcal{T}^2(\mathbf{J}) = -\frac{\mathbf{J}}{4} + \mathcal{K}^2(\mathbf{J}) \tag{19}$$

where $\mathcal{K}$ is the magnetic field integral equation operator

$$\mathcal{K}(\mathbf{J}) = \hat{\mathbf{n}} \times \int_S \nabla g(\mathbf{r}, \mathbf{r}') \times \mathbf{J} dS \tag{20}$$

Based on the discussion above, the solution of EFIE can be achieved by solving the preconditioned system

$$\mathcal{T}^2(\mathbf{J}) = \mathcal{T}\left(\hat{\mathbf{n}} \times \mathbf{E}^{inc}\right) \tag{21}$$

Assume that the inner and the outer operator are both discretized with $\mathbf{f}_n^{\text{loop}}$, $\mathbf{f}_n^{\text{flower}}$ and tested with $\hat{\mathbf{n}} \times \mathbf{f}_n^{\text{loop}}$, $\hat{\mathbf{n}} \times \mathbf{f}_n^{\text{flower}}$, then the corresponding Gram matrix $\bar{\bar{\mathbf{G}}}$ can be expressed by

$$\bar{\bar{\mathbf{G}}} = \begin{bmatrix} \bar{\bar{\mathbf{G}}}^{LL} & \bar{\bar{\mathbf{G}}}^{LF} \\ \bar{\bar{\mathbf{G}}}^{FL} & \bar{\bar{\mathbf{G}}}^{FF} \end{bmatrix} = \begin{bmatrix} \left\langle \hat{\mathbf{n}} \times \mathbf{f}^L, \mathbf{f}^L \right\rangle & \left\langle \hat{\mathbf{n}} \times \mathbf{f}^L, \mathbf{f}^F \right\rangle \\ \left\langle \hat{\mathbf{n}} \times \mathbf{f}^F, \mathbf{f}^L \right\rangle & \left\langle \hat{\mathbf{n}} \times \mathbf{f}^F, \mathbf{f}^F \right\rangle \end{bmatrix} \tag{22}$$

The preconditioned system can be rewritten as

$$\bar{\bar{\mathbf{Z}}}\bar{\bar{\mathbf{G}}}^{-1}\bar{\bar{\mathbf{Z}}} \cdot \bar{\mathbf{J}} = \bar{\bar{\mathbf{Z}}}\bar{\bar{\mathbf{G}}}^{-1} \cdot \bar{\mathbf{V}} \tag{23}$$

where $\bar{\bar{\mathbf{Z}}}$ is impendence matrix concerning with loop flower basis functions, $\bar{\mathbf{J}}$ is coefficient vector of current, $\bar{\mathbf{V}}$ is excitation vector. It is clear that $\bar{\bar{\mathbf{Z}}}\bar{\bar{\mathbf{G}}}^{-1}$ is the preconditioner of initial EFIE discretized system. The preconditioner introduced in (23) is rather simpler than previous ones, because the two matrices associated with the left and right operator are the same.

5. NUMERICAL RESULTS

In this section, the theoretical analysis of Gram matrices will be verified on a sphere discretized with quasi-uniform meshes of different mesh size h, and in addition, the effectiveness of loop flower basis functions will be demonstrated on a PEC conesphere.

5.1. Gram Matrices Associated with Loop-flower Basis Functions

We first analyze the conditioning of Gram matrices on a PEC sphere with a radius of 1 m. Fig. 3(a) shows that the condition number of the RWG Gram matrix maintains the same level versus the mesh size. It is clear to find that the standard RWG Gram matrix is well-conditioned. Similar to RWG basis functions, Fig. 3(b) shows that the condition number of the BC Gram matrix also remains very small, slightly bigger than that of RWG. Fig. 3(c) shows that the condition number of the loop-flower Gram matrix reduces against the increment of mesh size. It is clear that the condition number of the loop-flower Gram matrix has the theoretically predicted growth of $1/h^2$. Fig. 3(d) shows that the condition number of the Gram matrix of loop-flower basis functions in the Calderon preconditoner also has a growth of $1/h^2$.

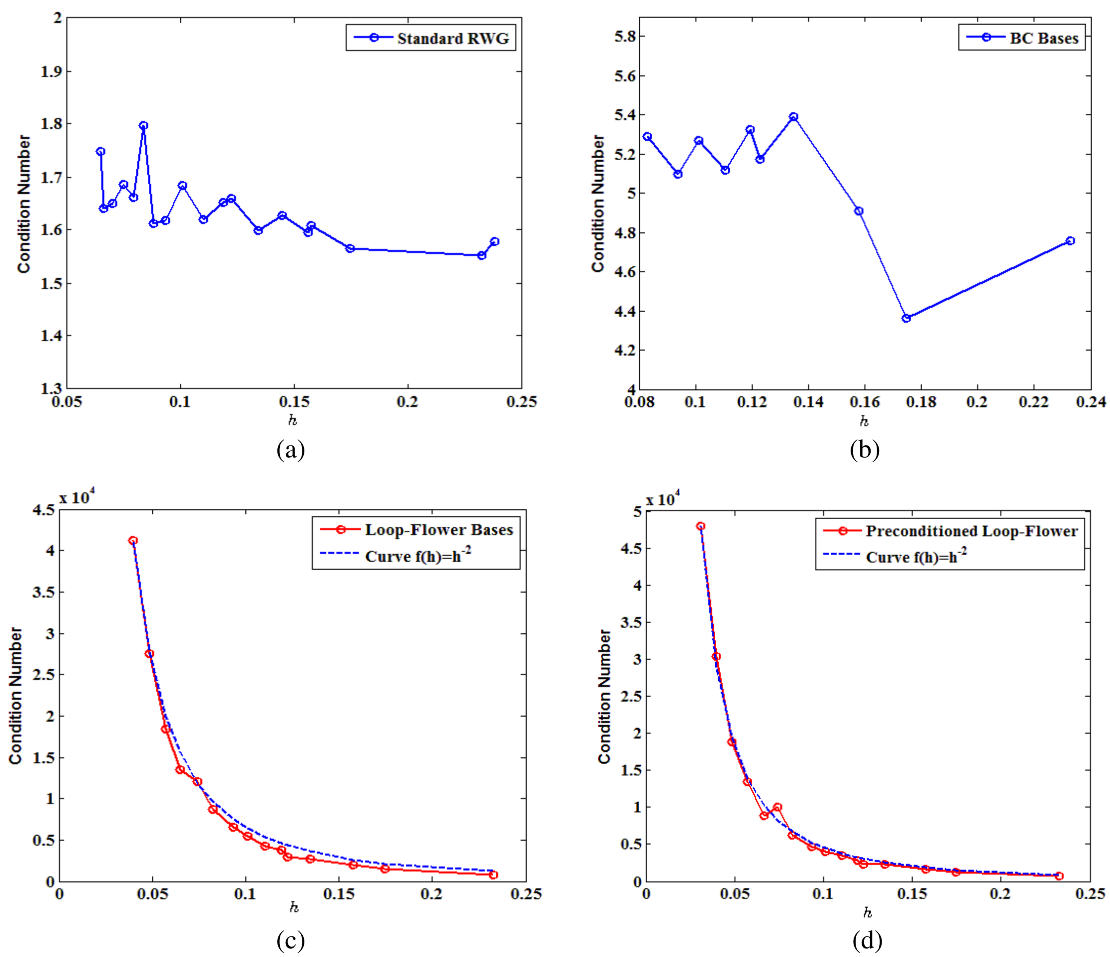

Figure 2: Condition Numbers of Gram matrices associated with different basis functions versus different values of mesh size. (a) Gram matrix of standard RWG basis functions. (b) Gram matrix of BC basis functions. (c) Gram matrix of loop-flower basis functions. (d) Gram matrix of loop-flower basis functions with Calderon preconditioner.

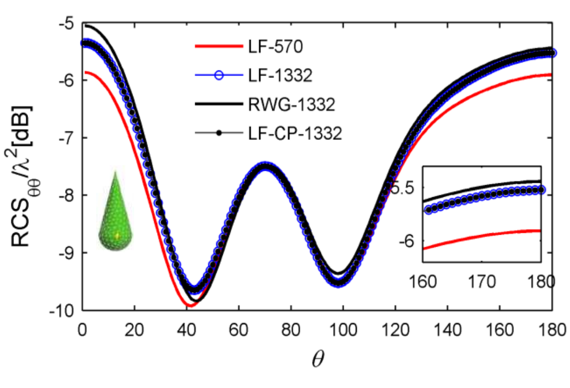

Figure 3: BiRCSs of the PEC cone-sphere at 0.3 GHz.

5.2. Calculate RCS Using Loop-flower Basis Functions

A PEC cone-sphere is analyzed as an example. The height of the cone is 1 m and the radius of the hemi-sphere is 0.25 m. The cone-sphere is analyzed twice on a coarse mesh (570 RWGs) and a fine mesh (1332 RWGs), respectively. The biRCSs are shown in Fig. 3. It can be seen that the accuracy can be improved using finer mesh structures.

In the case of PEC cone-sphere, there exists a sharp tip. The discrepancies between the results obtained by using loop-flower basis and those by using RWG basis or loop-star basis functions are larger than that in the PEC sphere with smooth surface. Careful examination shows that the currents near the sharp tip tend to have larger errors than currents on smooth surfaces. This can be explained from the fact that in a flower basis function, the currents are assumed to flow away from the reference node almost uniformly in all directions. However, at sharp edges, tips and corners, this approximation is acceptable only when the mesh structure is sufficiently fine.

6. CONCLUSION

In this paper, a theoretical analysis of the conditioning of loop-flower Gram matrix has been presented. The analysis shows that condition number of loop-flower Gram matrix has a growth of $1/h^2$. In addition, numerical example shows that loop-flower basis functions are effective for solving electromagnetic scattering problems.

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