# Propagation of TM Waves in a Double-layer Nonlinear Inhomogeneous Cylindrical Waveguide

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**Abstract**— The paper focuses on the problem of electromagnetic TM wave propagation in a two-layered circular cylindrical dielectric waveguide (this is an example of Bragg's waveguide) with a inhomogeneous nonlinear permittivity inside one of its layers, the other layer has a constant permittivity. The physical problem is reduced to a nonlinear transmission eigenvalue problem for Maxwell's equations. Eigenvalues of the problem correspond to propagation constants of the waveguide. Existence of eigenvalues of the considered problem is stated. Numerical experiments are carried out for three types of nonlinearities. Comparisons with linear/nonlinear inhomogeneous cases are given.

## 1. INTRODUCTION

Investigation of the spectra of electromagnetic waveguide systems is still an urgent task (see, e.g., [1–4]). Analysis of electromagnetic wave propagation (TE or TM) in a plane layer or in a cylindrical waveguide filled with nonlinear homogeneous medium is of special interest. One of the simplest cases in these problems refers to a single-layer waveguide in which the dielectric permittivity is described by the Kerr law (see, e.g., [4, 5]). Some waveguide problems for cylindrical waveguides are solved by methods of the theory of integral equations (see, for example, [4, 5]). The multilayer inhomogeneous nonlinear waveguide structure in the case of TE waves is studied in [7, 8]; the case of TM waves is studied numerically in [6].

## 2. STATEMENT OF THE PROBLEM

Consider three-dimensional space $\mathbb{R}^3$ with cylindrical coordinate system $O\rho\varphi z$. The space is filled with isotropic medium with constant permittivity $\varepsilon = \varepsilon_0\varepsilon_3$, where $\varepsilon_0 > 0$ is the permittivity of free space. In this medium a circular cylindrical waveguide

$$\Sigma := \{(\rho, \varphi, z) : 0 \leqslant \rho < R_1, 0 \leqslant \varphi < 2\pi\} \cup \{(\rho, \varphi, z) : R_1 \leqslant \rho \leqslant R_2, 0 \leqslant \varphi < 2\pi\}$$

is placed; its generating line (the waveguide axis) is parallel to the axis $Oz$. The waveguide is filled with isotropic nonmagnetic medium. Throughout the paper we assume that $\mu = \mu_0$, where $\mu_0 > 0$ is the permeability of free space. The waveguide is unbounded in $z$ direction (see Figure 1).

Consider the monochromatic TM waves $(\mathbf{E}, \mathbf{H})e^{-i\omega t}$, where $\mathbf{E} = (E_\rho, 0, E_z)^T$, $\mathbf{H} = (0, H_\varphi, 0)^T$, $\omega$ is the circular frequency, the quantities $\mathbf{E}$, $\mathbf{H}$ are called the complex amplitudes [1], $(\cdot)^T$ denotes the transpose operation, and components of the field $\mathbf{E}$, $\mathbf{H}$ do not depend on $\varphi$.

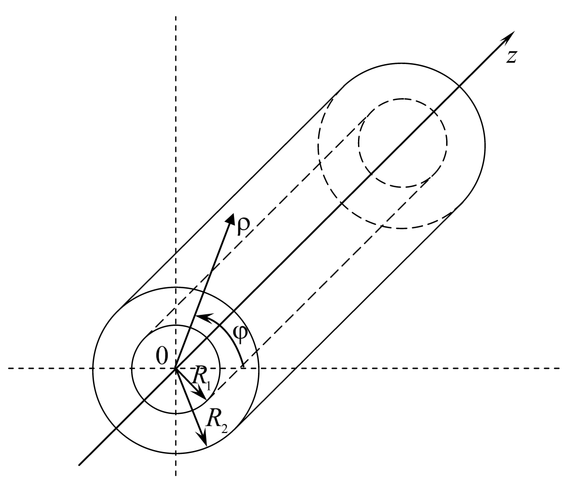


Figure 1: Geometry of the problem.

Complex amplitudes of the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ satisfy stationary Maxwell's equations

$$\operatorname{rot}\mathbf{H} = -i\omega\varepsilon\mathbf{E}, \quad \operatorname{rot}\mathbf{E} = i\omega\mu\mathbf{H}, \tag{1}$$

the continuity condition for the tangential components on the media interfaces (on the boundaries of the waveguide) $\rho = R_1$, $\rho = R_2$, and the radiation condition at infinity: the electromagnetic field decays as $O(\rho^{-1})$ when $\rho \to \infty$. The solutions to Maxwell's equations are sought in the entire space.

The dielectric permittivity in the whole space has the form $\varepsilon = \tilde{\varepsilon}\varepsilon_0$, where

$$\tilde{\varepsilon} = \begin{cases} \varepsilon_1, & 0 \leqslant \rho < R_1, \\ \varepsilon_2(\rho) + \alpha f\left(|\mathbf{E}|^2\right), & R_1 \leqslant \rho \leqslant R_2, \ \} \\ \varepsilon_3, & \rho > R_2, \end{cases} \tag{2}$$

and $\min\{\varepsilon_1, \varepsilon_3\} \geqslant \varepsilon_0 > 0$, $\alpha > 0$ are real constants, $\varepsilon_2(\rho) \in C^1[R_1, R_2]$, $f(u) \in C[0, +\infty)$.

We assume that the surface waves propagating along the axis $Oz$ of the waveguide $\Sigma$ depend harmonically on $z$. Thus, the components have the form

$$E_\rho = \mathrm{E}_\rho(\rho)e^{i\gamma z}, \quad E_z = \mathrm{E}_z(\rho)e^{i\gamma z}, \quad H_\varphi = \mathrm{H}_\varphi(\rho)e^{i\gamma z}, \tag{3}$$

where $\gamma$ is the (real) propagation constant (the spectral parameter of the problem).

Let $k_0^2 := \omega^2\mu_0\varepsilon_0$. Substituting the complex amplitudes $\mathbf{E}$ and $\mathbf{H}$ with components (3) into Equation (1) and introducing the notation $u_1(\rho;\gamma) := \mathrm{E}_\rho(\rho;\gamma)$, $u_2(\rho;\gamma) := i\mathrm{E}_z(\rho;\gamma)$ we obtain

$$\gamma u_2' + \left(\gamma^2 - k_0^2\tilde{\varepsilon}\right)u_1 = 0, \quad -\gamma\rho^{-1}(\rho u_1)' - \rho^{-1}\left(\rho u_2'\right)' - k_0^2\tilde{\varepsilon}u_2 = 0, \tag{4}$$

where $u_1$, $u_2$ are real functions and $\tilde{\varepsilon}$ is given by (2). Also we assume that the functions $u_1$, $u_2$ are sufficiently smooth.

Tangential components of electromagnetic field are known to be continuous at media interfaces. In this case the tangential components are $E_z$ and $H_\varphi$. Taking into account that $H_\varphi(\rho) = \frac{1}{i\omega\mu}(i\gamma E_\rho(\rho) - E_z'(\rho))$, we obtain

$$[\gamma u_1 + u_2']\big|_{\rho=R_1} = 0, \quad [\gamma u_1 + u_2']\big|_{\rho=R_2} = 0, \quad [u_2]\big|_{\rho=R_1} = 0, \quad [u_2]\big|_{\rho=R_2} = 0, \tag{5}$$

where $[v]\big|_{\rho=s} = \lim_{\rho\to s-0} v(\rho) - \lim_{\rho\to s+0} v(\rho)$.

*Problem $P_M$ is to find quantities $\gamma = \widehat{\gamma}$ for which a non-trivial solution $u_1(\rho;\widehat{\gamma})$, $u_2(\rho;\widehat{\gamma})$ to (4)–(5) exists; this solutions has prescribed value $u_1(R_1)$, $u_2(R_1)$ (or $u_1(R_2)$, $u_2(R_2)$) and decays as $O(\rho^{-1})$ when $\rho \to \infty$. The quantities solving problem $P_M$ are called eigenvalues, the corresponding functions $u_1(\rho;\widehat{\gamma})$, $u_2(\rho;\widehat{\gamma})$ are called eigenfunctions.*

**Statement 1.** *Let $\min\{\varepsilon_1, \varepsilon_3\} \geqslant \varepsilon_0$, $\alpha > 0$ be real constants and the linear problem (Problem $P_M$ for $\alpha = 0$) have $k \geqslant 1$ solutions $\tilde{\gamma}_i$, $i = \overline{1,k}$. Then there is a value $\alpha_0 > 0$ such that, for any $\alpha \leqslant \alpha_0$, problem $P_M$ has at least one eigenvalue in the vicinity of each $\tilde{\gamma}_i$.*

It follows from this statement that there exist axial-symmetric $TM$-polarized waves propagating without attenuation in inhomogeneous cylindrical dielectric waveguides of circular cross section filled with non-magnetic isotropic inhomogeneous nonlinear medium.

## 3. NUMERICAL RESULTS

Numerical experiments are carried out for the following functions:

1. $\varepsilon = \varepsilon(\rho) + \alpha u^2$, $\alpha = 0.01$, where (a) $\varepsilon(\rho) = \varepsilon_2 + \rho^{-1}$, (b) $\varepsilon(\rho) = \varepsilon_2 + \rho$;
2. $\varepsilon = \varepsilon(\rho) + \frac{\alpha u^2}{1+\beta u^2}$, $\alpha = 0.01$, $\beta = 0.001$, where (a) $\varepsilon(\rho) = \varepsilon_2 + \rho^{-1}$, (b) $\varepsilon(\rho) = \varepsilon_2 + \rho$;
3. $\varepsilon = \varepsilon(\rho) + \alpha\left(1 - e^{-\beta u^2}\right)$, $\alpha = 5$, $\beta = 0.01$, where (a) $\varepsilon(\rho) = \varepsilon_2 + \rho^{-1}$, (b) $\varepsilon(\rho) = \varepsilon_2 + \rho$,

$\varepsilon_2$ is a positive real constant.

The following values of parameters are used for calculations given in Figures 2–19: $\varepsilon_1 = 4$, $\varepsilon_2 = 9$, $\varepsilon_3 = 1$, $R_1 = 2$, $2 < R_2 < 8$, $k_0 = 1$. Initial conditions for carrying out the calculations are defined in [6], here we also use $C_1 = 1$.

In this report the dependence $\gamma(\Delta R)$ is called a dispersion curve, where $\Delta R = R_2 - R_1$, $R_1$ is fixed, $R_2$ is varied. Dispersion curves for the cases 1(a) and 1(b), 2(a) and 2(b), and 3(a) and 3(b) are shown in Figs. 2 and 3, 8 and 9, and 14 and 15, respectively. The vertical axis corresponds to $\gamma^2 > 4$ and the horizontal axis corresponds to the thickness $0 < \Delta R = R_2 - R_1 < 6$ of the outer cylindrical shell (see Figure 1). The dotted red curves correspond to the linear inhomogeneous case and the blue dotted curves correspond to the nonlinear inhomogeneous case (in the same figure the inhomogeneity is the same for both types of curves; red and blue curves differ in a nonlinear term only).

The vertical dashed lines in Figs. 2 and 8, 3 and 9, 14 and 15 correspond to $\Delta R = 1.5$, $\Delta R = 1.2$, $\Delta R = 1.242$ and $\Delta R = 1.075$, respectively. Eigenvalues of the problem are points of intersections of these dashed lines with the dispersion curves; first three points in Figures 2 and 3 are marked (the smallest value, red dot, corresponds to the linear case and the others two (green and black dots) correspond to the Kerr case); first four points in Figures 8, 9, 14, 15 are marked (the smallest value, red dot, corresponds to the linear case and the others three (green, blue and black dots) correspond to the case with saturated nonlinearity).

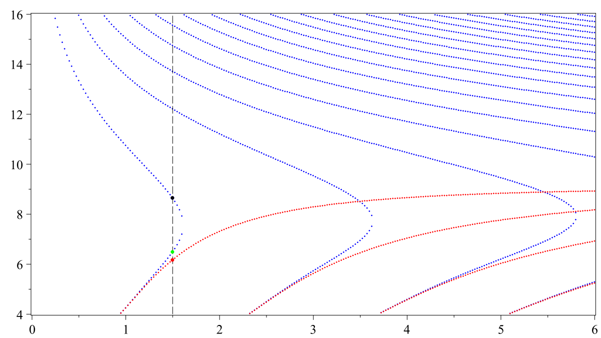


Figure 2: Case 1(a). Marked eigenvalues: $\gamma \approx 2.482$ (red dot), $\gamma \approx 2.546$ (green dot), and $\gamma \approx 2.940$ (black dot).

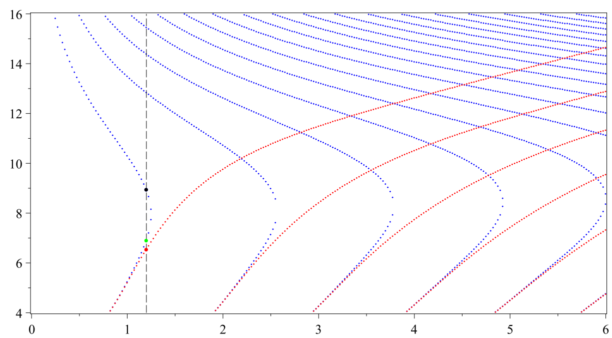


Figure 3: Case 1(b). Marked eigenvalues: $\gamma \approx 2.554$ (red dot), $\gamma \approx 2.624$ (green dot), and $\gamma \approx 3.988$ (black dot).

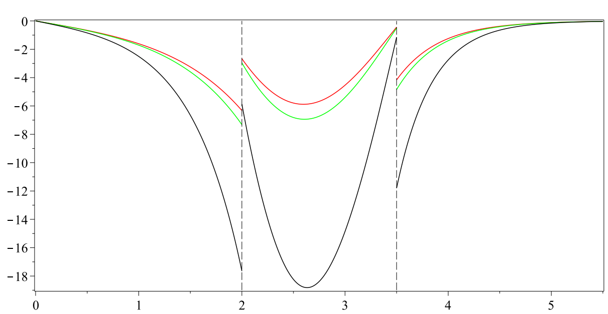


Figure 4: Eigenfunction $u_1(\rho)$ for the case 1(a): $R_2 = 3.5$; $\gamma \approx 2.482$ (red), $\gamma \approx 2.546$ (green), and $\gamma \approx 2.940$ (black).

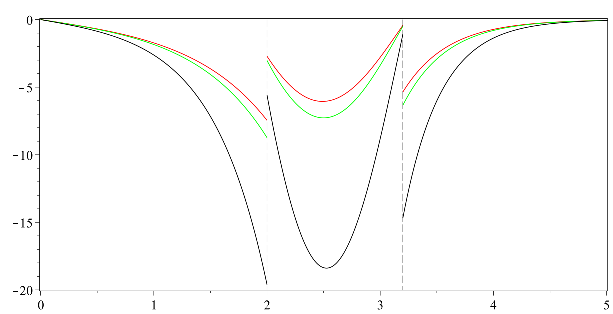


Figure 5: Eigenfunctions $u_1(\rho)$ for the case 1(b): $R_2 = 3.2$; $\gamma \approx 2.554$ (red), $\gamma \approx 2.624$ (green), and $\gamma \approx 2.988$ (black).

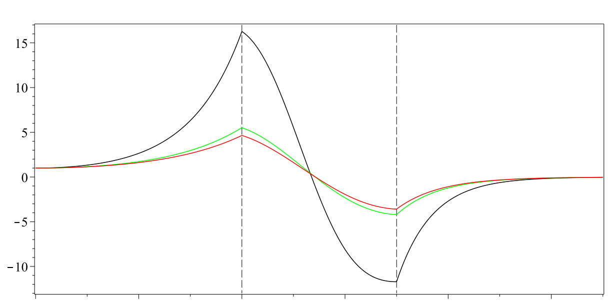


Figure 6: Eigenfunctions $u_2(\rho)$ for the case 1(a): $R_2 = 3.5$; $\gamma \approx 2.482$ (red), $\gamma \approx 2.546$ (green), and $\gamma \approx 2.940$ (black).

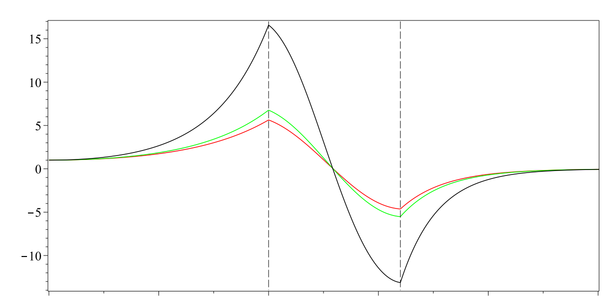


Figure 7: Eigenfunctions $u_2(\rho)$ for case 1(b): $R_2 = 3.2$; $\gamma \approx 2.554$ (red), $\gamma \approx 2.624$ (green), and $\gamma \approx 2.988$ (black).

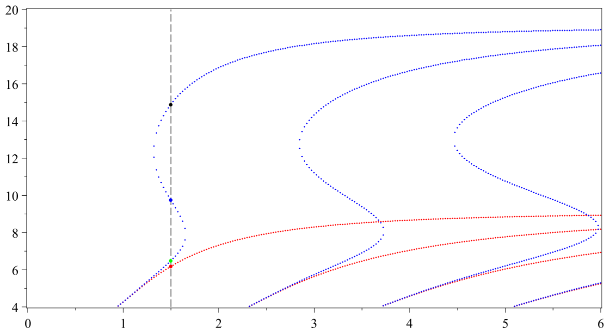

Figure 8: Case 2(a). Marked eigenvalues: $\gamma \approx 2.482$ (red dot), $\gamma \approx 2.5425$ (green dot), $\gamma \approx 3.12$ (blue dot), and $\gamma \approx 3.855$ (black dot).

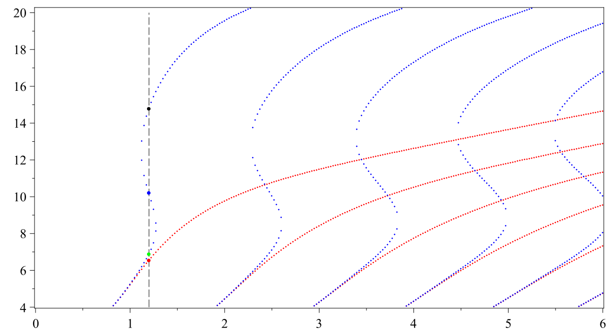

Figure 9: Case 2(b). Marked eigenvalues: $\gamma \approx 2.554$ (red dot), $\gamma \approx 2.62$ (green dot), $\gamma \approx 3.1925$ (blue dot), and $\gamma \approx 3.8425$ (black dot).

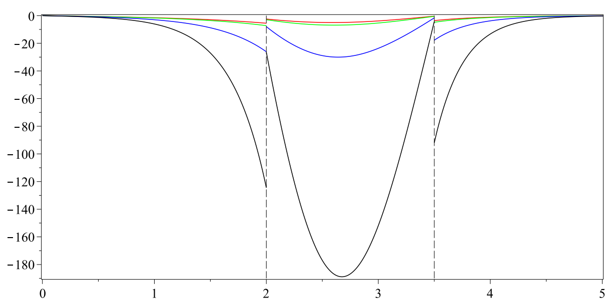

Figure 10: Eigenfunctions $u_1(\rho)$ for the case 2(a): $R_2 = 3.5$; $\gamma \approx 2.482$ (red), $\gamma \approx 2.5425$ (green), $\gamma \approx 3.12$ (blue), and $\gamma \approx 3.855$ (black).

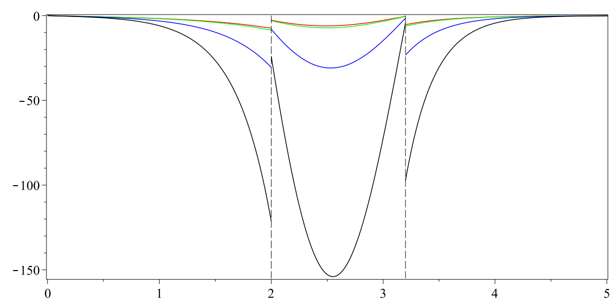

Figure 11: Eigenfunctions $u_1(\rho)$ for the case 2(b): $R_2 = 3.2$; $\gamma \approx 2.554$ (red), $\gamma \approx 2.62$ (green), $\gamma \approx 3.1925$ (blue), and $\gamma \approx 3.8425$ (black).

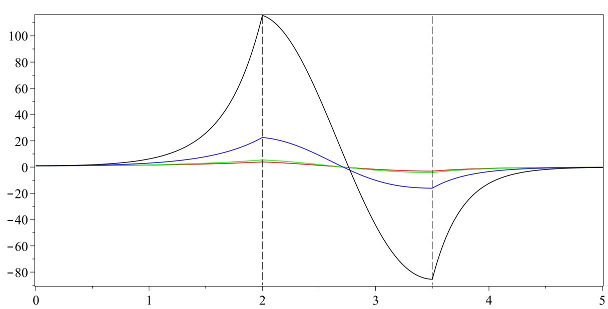

Figure 12: Eigenfunctions $u_2(\rho)$ for the case 2(a): $R_2 = 3.5$; $\gamma \approx 2.482$ (red), $\gamma \approx 2.5425$ (green), $\gamma \approx 3.12$ (blue), and $\gamma \approx 3.855$ (black).

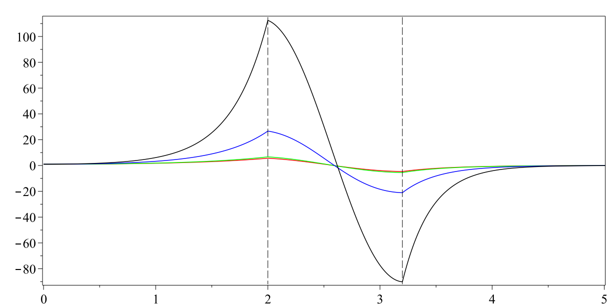

Figure 13: Eigenfunctions $u_2(\rho)$ for the case 2(b): $R_2 = 3.2$; $\gamma \approx 2.554$ (red), $\gamma \approx 2.62$ (green), $\gamma \approx 3.1925$ (blue), and $\gamma \approx 3.8425$ (black).

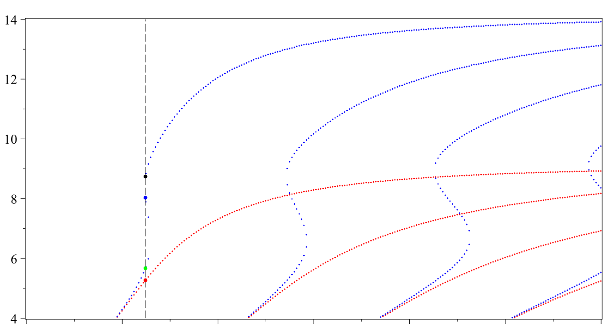

Figure 14: Case 3(a). Marked eigenvalues: $\gamma \approx 2.294$ (red dot), $\gamma \approx 2.38$ (green dot), $\gamma \approx 2.8325$ (blue dot), and $\gamma \approx 3.955$ (black dot).

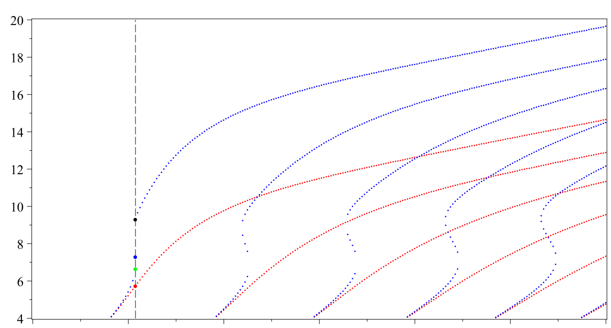

Figure 15: Case 3(b). Marked eigenvalues: $\gamma \approx 2.388$ (red dot), $\gamma \approx 2.5725$ (green), $\gamma \approx 2.695$ (blue), and $\gamma \approx 3.045$ (black dot).

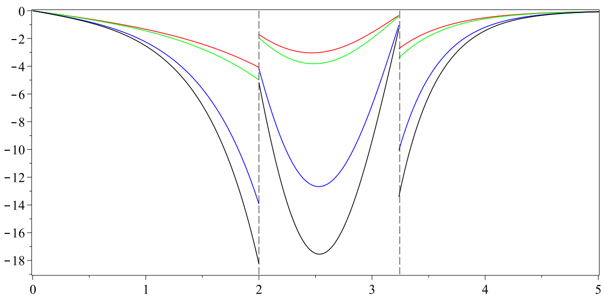

Figure 16: Eigenfunctions $u_1(\rho)$ for the case 3(a): $R_2 = 3.245$; $\gamma \approx 2.294$ (red), $\gamma \approx 2.38$ (green), $\gamma \approx 2.8325$ (blue), and $\gamma \approx 2.955$ (black).

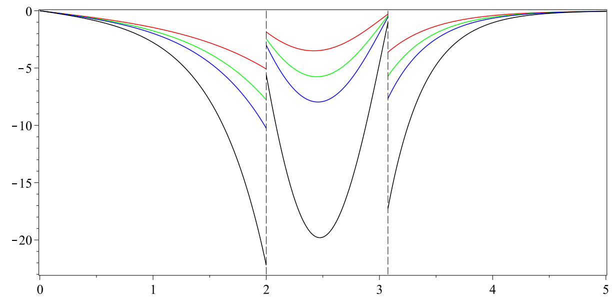

Figure 17: Eigenfunctions $u_1(\rho)$ for the case 3(b): $R_2 = 3.075$; $\gamma \approx 2.388$ (red), $\gamma \approx 2.5725$ (green), $\gamma \approx 2.695$ (blue), and $\gamma \approx 3.045$ (black).

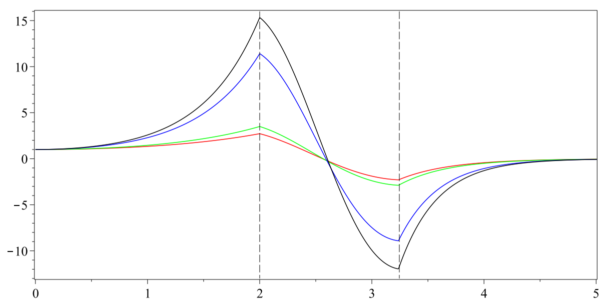

Figure 18: Eigenfunctions $u_2(\rho)$ for the case 3(a): $R_2 = 3.245$; $\gamma \approx 2.294$ (red), $\gamma \approx 2.38$ (green), $\gamma \approx 2.8325$ (blue), and $\gamma \approx 2.955$ (black).

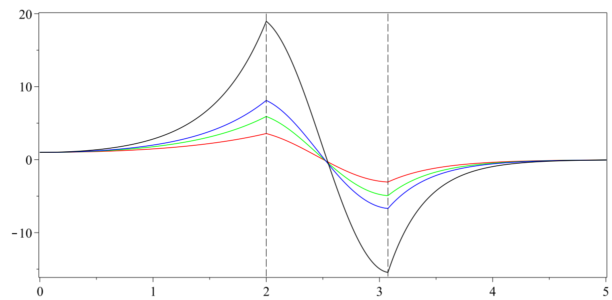

Figure 19: Eigenfunctions $u_2(\rho)$ for the case 3(b): $R_2 = 3.075$; $\gamma \approx 2.388$ (red), $\gamma \approx 2.5725$ (green), $\gamma \approx 2.695$ (blue), and $\gamma \approx 3.045$ (black).

In Figures 4–7, 10–13, and 16–19, eigenfunctions $u_1(\rho)$, $u_2(\rho)$ are plotted for the cases 1–3, respectively. The vertical axis corresponds to the value of the plotted functions $u_1(\rho)$, $u_2(\rho)$ and the horizontal axis corresponds to the value $\rho$.

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