Resonant Properties of HE_{111} Mode of a Complicated Microwave Cavity for a New Type of Rubidium Clock

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Abstract— Resonant properties of HE_{111} mode of a complicated microwave cavity with ceramic material, used in rubidium clock, are studied by mode matching method. The microwave cavity works at a certain frequency such as 6835 GHz by accommodating a glass bubble containing rubidium vapor. To make electromagnetism focus on the centre of the cavity for energy exchange and further miniaturize the cavity to a large extent, a ceramic dielectric ring is installed in the inner layer of the glass bubble. In order to study main factors influencing resonant characteristics, resonant frequencies of HE_{111} mode are calculated through eigen equation and compared with simulated results. The results show that theoretical computations are in good agreement with finite element simulations. In addition, the effects of loaded dielectric and rubidium vapor on resonant frequency are also analyzed. This work is of great significance for the miniaturizing of the cavity and theory perfection in atomic clock.

1. INTRODUCTION

Rubidium frequency standard is widely used in global positioning system (GPS) [1], communication, navigation, positioning and running survey system for its good properties such as small volume, strong environment adaptability and good frequency flexibility [2, 3]. One of the most important microwave components in it is microwave cavity, which directly plays an impact on the properties of atom frequency standard [4, 5]. Placing a glass bubble filled with rubidium gas in the microwave cavity, microwave radiation field provided by microwave cavity can interact with rubidium gas inside the glass bubble and make rubidium atomic level transition occur [1, 6]. For some applications, it is desirable that microwave radiation field changes with the energy of rubidium gas at a certain frequency such as 6.835 GHz [7]. Generally, the resonant frequency is related to material property and rubidium gas parameters. So it is extremely critical for rubidium frequency standard design because of the effect of material property and rubidium gas parameters on resonant frequency.

The eigen value problem of microwave cavity filled with complicated materials belongs to the complex boundary value problem of electric and magnetic fields. And many methods, such as variation method, difference method, finite element method and mode matching method, etc. [8–13], are employed to study this problem. Though most researchers have studied boundary value problem of electric and magnetic fields, few papers published were paid more attention on theoretical analysis of complicated microwave cavity filled with rubidium gas [14, 15].

In this paper, the resonant frequency of HE_{111} mode of the microwave cavity filled with ceramic material and rubidium gas is studied by mode matching method. In addition, the effects of the ceramic material and rubidium gas on the resonant frequency of the HE_{111} mode are also analyzed. In addition, this work will propose a theoretical basis for the miniaturization of the microwave cavity.

2. THEORETICAL ANALYSIS

The basic structure of complicated microwave cavity which is used to exchange energy between gaseous rubidium and electromagnetic fields in rubidium frequency standard is modeled in Fig. 1. It is divided into three radial regions: p, q and t. Among these regions, region 2 will be filled with gaseous rubidium, regions 1, 3, 5, 7 and 9 are enclosed glass bubble, and region 6 is filled with ceramics. The microwave cavity and the outmost circuit are joined by regions 4, 8 and 10. The medium parameters in these regions are characterized by μ_{ri} and ε_{ri} (i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10), respectively, where μ_{ri} is the relative permeability of part i and ε_{ri} is the relative permittivity. All the loaded dielectrics are isotropic and $\mu_{r1} = \mu_{r2} = \mu_{r3} = \mu_{r4} = \mu_{r5} = \mu_{r6} = \mu_{r7} = \mu_{r8} = 1$.

In cylindrical coordinates, the longitudinal electric field components H_z of TE_{011} mode in region

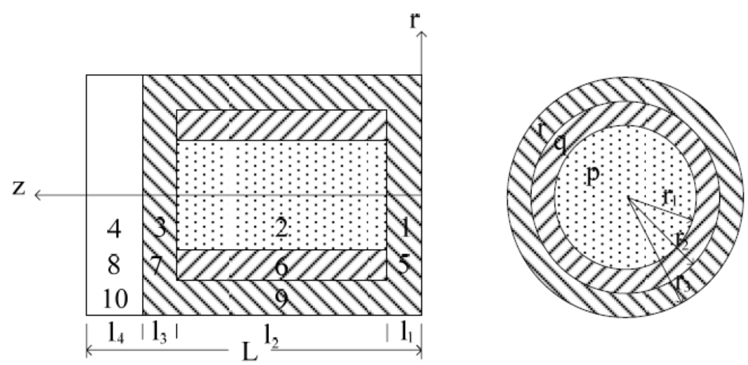

Figure 1: Basic structure of microwave cavity.

p are expressed as

$$H_{zp} = \sum_{m=1}^{p} A_m^p J_1 \left(k_m^p r\right) \Phi_m^p(z) \quad \text{(region } p\text{)}, \tag{1}$$

$$H_{zq} = \sum_{m=1}^{\infty} \left[A_m^q J_1 \left(k_m^q r\right) + B_m^q N_1 \left(k_m^q r\right)\right] \Phi_m^q(z) \quad \text{(region } q\text{)}, \tag{2}$$

Considering the boundary condition of vanishing normal magnetic fields on the metallic wall surface, the magnetic field along the z axis in region t is

$$H_{zt} = \sum_{m=1}^{\infty} A_m^t \left[J_1 \left(k_m^t r\right) + \frac{J_1 \left(k_m^t r_3\right)}{N_1' \left(k_m^t r_3\right)} N_1 \left(k_m^t r\right) \right] \Phi_m^t(z) \quad \text{(region } t\text{)}, \tag{3}$$

where $J_1(k_m r)$ and $N_1(k_m r)$ are the first order Bessel function and zero order Neuman function, respectively. k_m^p, k_m^q and k_m^q represent the transverse propagation constants of region p, q and t, respectively. They are related to the resonant wave number k_0 and propagation constant β_{mi} in the following manner

$$\begin{aligned} \left(k_m^p\right)^2 &= k_0^2 \varepsilon_{ri} - \beta_{mi}^2 \, (i = 1, 2, 3, 4), \quad \left(k_m^q\right)^2 = k_0^2 \varepsilon_{ri} - \beta_{mi}^2 \, (i = 5, 6, 7, 8), \\ \left(k_m^t\right)^2 &= k_0^2 \varepsilon_{ri} - \beta_{mi}^2 \, (i = 9, 10). \end{aligned} \tag{4}$$

In region p, $H_z|_{z=0,L} = 0$. Enforcing field continuity condition of E_φ and H_r at the interface between the dielectric layers, the following Eigen equation comes into existence

$$\begin{aligned} &\frac{\tan \beta_{m2} l_2}{\beta_{m2}} + \frac{\tan \beta_{m1} l_1}{\beta_{m1}} + \frac{\tan \beta_{m3} l_3}{\beta_{m3}} + \frac{\tan \beta_{m4} l_4}{\beta_{m4}} - \frac{\beta_{m3}}{\beta_{m2}\beta_{m4}} \tan \beta_{m2} l_2 \tan \beta_{m3} l_3 \tan \beta_{m4} l_4 \\ &- \frac{\beta_{m3}}{\beta_{m1}\beta_{m4}} \tan \beta_{m1} l_1 \tan \beta_{m3} l_3 \tan \beta_{m4} l_4 - \frac{\beta_{m2}}{\beta_{m1}\beta_{m3}} \tan \beta_{m1} l_1 \tan \beta_{m2} l_2 \tan \beta_{m3} l_3 \\ &- \frac{\beta_{m2}}{\beta_{m1}\beta_{m4}} \tan \beta_{m1} l_1 \tan \beta_{m2} l_2 \tan \beta_{m4} l_4 = 0 \end{aligned} \tag{5}$$

Substituting (4) into (5), an equation about k_{mp} and k_0 is obtained.

In the same way, considering boundary condition $H_z|_{z=0,L} = 0$ and continuity condition of E_φ

and H_r lead to the following Eigen equation in region q and t

$$\begin{aligned}&\frac{\tan\beta_{m6}l_2}{\beta_{m6}}+\frac{\tan\beta_{m5}l_1}{\beta_{m5}}+\frac{\tan\beta_{m7}l_3}{\beta_{m7}}+\frac{\tan\beta_{m8}l_4}{\beta_{m8}}-\frac{\beta_{m7}}{\beta_{m6}\beta_{m8}}\tan\beta_{m6}l_2\tan\beta_{m7}l_3\tan\beta_{m8}l_4\\&-\frac{\beta_{m7}}{\beta_{m5}\beta_{m8}}\tan\beta_{m5}l_1\tan\beta_{m7}l_3\tan\beta_{m8}l_4-\frac{\beta_{m6}}{\beta_{m5}\beta_{m7}}\tan\beta_{m5}l_1\tan\beta_{m6}l_2\tan\beta_{m7}l_3\\&-\frac{\beta_{m6}}{\beta_{m5}\beta_{m8}}\tan\beta_{m5}l_1\tan\beta_{m6}l_2\tan\beta_{m8}l_4=0,\end{aligned}\tag{6}$$

$$\frac{\tan\beta_{m7}\left(l_1+l_2+l_3\right)}{\beta_{m7}}+\frac{\tan\beta_{m8}l_4}{\beta_{m8}}=0\tag{7}$$

By matching the tangential field components at $r = r_1$ and $r = r_2$ and using the orthogonality of trigonometric function, a linear homogeneous system of equations is obtained as follows

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=0,\tag{8}$$

The special condition for the solution to these equations is that the determinant of the system matrix is equal to zero, i.e.,

$$\begin{vmatrix} X_1 & X_2 \\ X_3 & X_4 \end{vmatrix}=0.\tag{9}$$

where $A = [A_1^q, A_2^q, \ldots, A_m^q, \ldots]'$, $B = [B_1^q, B_2^q, \ldots, B_m^q, \ldots]'$, X_1, X_2, X_3 and X_4 are partitioned matrix, respectively. The elements of this matrix are

$$\begin{aligned}x_{1im}&=\int_0^L\Phi_m^q(z)\Phi_i^p(z)dz\left[\frac{J_1\left(k_m^q r_1\right)}{J_1\left(k_i^q r_1\right)}-\frac{J_1'\left(k_m^q r_1\right)k_i^p}{J_1'\left(k_i^p r_1\right)k_m^q}\right],\\x_{1im}&=\int_0^L\Phi_m^q(z)\Phi_i^p(z)dz\left[\frac{N_1\left(k_m^q r_1\right)}{J_1\left(k_i^p r_1\right)}-\frac{N_1'\left(k_m^q r_1\right)k_i^p}{J_1'\left(k_i^p r_1\right)k_m^q}\right],\\x_{3im}&=\int_0^L\Phi_m^q(z)\Phi_i^t(z)dz\left[\frac{J_1\left(k_m^q r_2\right)}{J_1\left(k_i^t r_2\right)-\frac{J_1'(k_i^t r_3)}{N_1'(k_i^t r_3)}N_1\left(k_i^t r_2\right)}-\frac{J_1'\left(k_m^q r_2\right)k_i^r}{k_m^q\left[J_1'\left(k_i^t r_2\right)-\frac{J_1'(k_i^t r_3)}{N_1'(k_i^t r_3)}N_1'\left(k_i^t r_2\right)\right]}\right],\\x_{4im}&=\int_0^L\Phi_m^q(z)\Phi_i^t(z)dz\left[\frac{N_1\left(k_m^q r_2\right)}{J_1\left(k_i^t r_2\right)-\frac{J_1'(k_i^t r_3)}{N_1'(k_i^t r_3)}N_1\left(k_i^t r_2\right)}-\frac{N_1'\left(k_m^q r_2\right)k_i^r}{k_m^q\left[J_1'\left(k_i^t r_2\right)-\frac{J_1'(k_i^t r_3)}{N_1'(k_i^t r_3)}N_1'\left(k_i^t r_2\right)\right]}\right].\end{aligned}$$

Resonant frequencies of TE_{011} mode of the complicated cavity can be calculated from (5), (6), (7) and (9).

3. CALCULATED RESULTS

In the microwave cavity, higher order modes have little influence on field distribution. In other words, the resonant properties are mainly determined by the low order modes. To make computation more convenient, (9) can be simplified to an identity by reserving the lowest mode.

Table 1 shows the frequencies of HE_{111} mode of the complicated microwave cavity with different medium permittivity, assuming dimensions of $l_1 = 2$ mm, $l_2 = 28$ mm, $l_3 = 2$ mm, $l_4 = 10$ mm, $r_1 = 28$ mm, $r_2 = 30$ mm, $r_3 = 32$ mm. f_1 and f_2 denote the computed results and simulated results, respectively, and the unit of frequency is GHz. Rubidium vapor can be seen as some kind of dielectric and its relative permittivity is about determined by the equation $\varepsilon_r \approx 1 + \frac{N}{\varepsilon_0}\frac{\mu_0^2}{3kT}$, $N = 10^5$, $T = 300$ K. So the relative permittivity of the rubidium vapor is about 1.0005. It is clearly seen that the computed results excellently agree with the simulated results.

Figure 2 shows the curves of resonant frequency varying with rubidium molecular number inside the cavity with the parameters of $l_1 = 2$ mm, $l_2 = 28$ mm, $l_3 = 2$ mm, $r_1 = 28$ mm, $r_2 = 30$, $r_3 = 32$ mm, $\varepsilon_{r1} = \varepsilon_{r3} = \varepsilon_{r4} = \varepsilon_{r5} = \varepsilon_{r8} = \varepsilon_{r9} = 2.8$ and $\varepsilon_{r4} = \varepsilon_{r8} = \varepsilon_{r10} = 1$. Parameter L denotes the total cavity length. For the complicated microwave cavity, the temperature of rubidium gas is 300 K. The results show that the resonant frequency of the complicated microwave

Table 1: The comparison between computed results and simulated results.

ε_{r1}	ε_{r2}	ε_{r3}	ε_{r4}	ε_{r5}	ε_{r6}	ε_{r7}	ε_{r8}	ε_{r9}	ε_{r10}	f_1	f_2
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	4.5060	4.4980
1.0000	1.0005	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	4.5050	4.4970
2.8000	1.0005	2.8000	1.0000	2.8000	9.8000	1.0000	2.8000	1.0000	1.0000	4.0860	3.9940
3.3000	1.0005	3.3000	1.0000	3.3000	9.8000	3.3000	1.0000	3.3000	1.0000	4.0720	3.9330
1.0000	1.0005	1.0000	1.0000	1.0000	9.8000	1.0000	1.0000	1.0000	1.0000	4.3690	4.2950

cavity gradually decreases with the increasing of rubidium molecules number. When the rubidium molecules number increases, the rubidium gas polarization enhances and the equivalent permittivity of the rubidium gas increases, which directly leads to slightly decrease of the resonant frequency. Besides, the length of the cavity may be another factor that might contributes to the resonant frequency variation.

Figure 3 indicates how the resonant frequency varies with the glass radius r_3. The parameters are set to be $l_1 = 2\,\mathrm{mm}$, $l_2 = 28\,\mathrm{mm}$, $l_3 = 2\,\mathrm{mm}$, $r_1 = 28\,\mathrm{mm}$, $r_2 = 30$, $\varepsilon_{r1} = \varepsilon_{r3} = \varepsilon_{r5} = \varepsilon_{r7} = \varepsilon_{r9} = 2.2$ and $\varepsilon_{r4} = \varepsilon_{r8} = \varepsilon_{r10} = 1$. As shown in Fig. 3, resonant frequency exhibits slightly decrease when the out radius r_3 increases. This phenomenon can be explained by dielectric perturbation theory [16]. As the outer radius increases, the perturbation body becomes larger, which directly causes the decrease of the frequency.

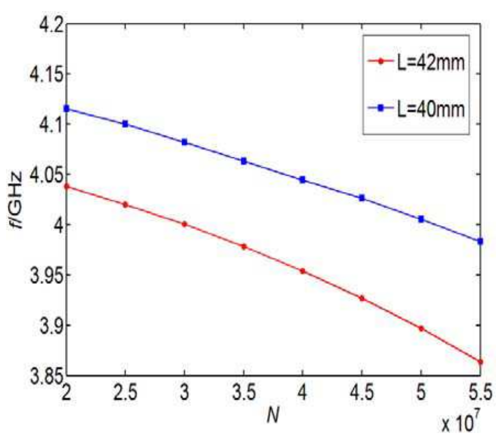

Figure 2: Variation of resonant frequency with molecule number.

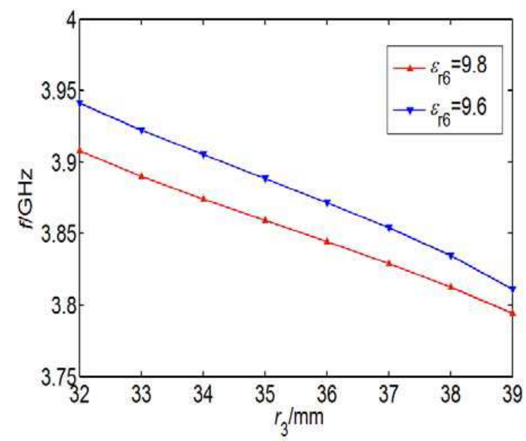

Figure 3: Variation of resonant frequency with r_3.

4. CONCLUSIONS

Resonant properties of a microwave cavity with ceramic material for a new type of rubidium clock are studied quantitatively by mode matching method. Resonant frequency of HE_{111} mode is then calculated by considering dielectric permittivity and rubidium gas parameters. The results show that decrease of the outer radius and the number of rubidium molecular number are all benefit for the increase of resonant frequency of the complicated microwave cavity with ceramic layer. These results play an important role in miniaturization and optimization of the microwave cavity for a new type of rubidium clock that has drawn considerably attention because of its potential application for communications.

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