Nonparaxial Propagation of Complex Variable Function Cosh-Gaussian Beams

Dongmei Deng[1, 2], **Chidao Chen**[1], **Yushan Zheng**[3], **Xi Peng**[1], **Bo Chen**[1], **Yulian Peng**[1], **and Meiling Zhou**[1]

[1]Laboratory of Nanophotonic Functional Materials and Devices
South China Normal University, Guangzhou 510631, China
[2]CAS Key Laboratory of Geospace Environment
University of Science and Technology of China, Chinese Academy of Sciences, Hefei 230026, China
[3]Shenzhen Entry-exit Inspection and Quarantine Bureau, Shenzhen 518045, China

Abstract— By applying the superposition of beams, we obtain a group of virtual sources for a complex variable function (CVF) Cosh-Gaussian wave. We also derive a closed-form expression for the CVF-Cosh-Gaussian wave that in the appropriate limit yielding the paraxial CVF-Cosh-Gaussian beam. From the spectral representation of the CVF-Cosh-Gaussian waves, we derive the jth-order nonparaxial corrections for the corresponding paraxial CVF-Cosh-Gaussian beam analytically. The angular velocity of the nonparaxial CVF-Cosh-Gaussian beams decreases with the transmission distance increasing.

1. INTRODUCTION

In recent years, the study of spiral light beams [1–8] has drawn much attention is due to their potential application in different optic technologies including the laser manipulation [6]. The spiral beams [4–11] which have singularities and are characterized by a non-zero orbital momentum, their structure of the intensity distribution are rotated during propagation.

In 1971, Deschamps [12] has presented the virtual source method. In 1976 and 1977, Felsen and his collaborators [13, 14] have systematically developed this method. By using this method, Seshadri [15–17] has derived the virtual sources of Bessel-Gaussian, elegant Hermite-Gaussian, and cylindrical symmetric elegant Laguerre-Gaussian waves. Bandres et al. [18] have introduced the higher-order complex source that generates elegant Laguerre-Gaussian waves with radial mode number n and angular mode number m in 2004. In 2007, Zhang et al. [19] have employed this method to study the propagation properties of cosh-Gaussian beams in and beyond the paraxial regime. We have obtained the virtual sources of Elegant Hermite-Laguerrel Gaussian beam [20], Hollow gaussian beam [21], and Pearcey beam [22]. In 2012, Yan et al. [23] have studied the virtual source for an Airy beam. However, the virtual sources of the complex variable function (CVF) Cosh-Gaussian beams [10, 11] have not been reported.

In this paper, based on the superposition of beams, a group of virtual sources that generate the scalar CVF-Cosh-Gaussian wave are identified. The integral and the differential representation for the CVF-Cosh-Gaussian waves have been derived. From the spectral representation of the CVF-Cosh-Gaussian waves, we obtain the exact nonparaxial solution and the nth-order nonparaxial corrections for the corresponding paraxial CVF-Cosh-Gaussian beams analytically.

2. VIRTUAL SOURCE OF CVF-COSH-GAUSSIAN BEAMS

Suppose $E(r, z)$ be a monochromatic scalar wave function that describes a CVF-Cosh-Gaussian wave propagating along the positive z axis. Function $E(r, z)$ satisfies the homogeneous Helmholtz equation for $z > 0$. The CVF-Cosh-Gaussian field is assumed to have the general form

$$E(x, y, z) = q_0/q(z)\cosh(\zeta_\pm)\exp(ikz)\exp(v_1), \tag{1}$$

where $q(z) = z - iz_R$, $q_0 = -iz_R$, $\zeta_\pm = (x \pm iy)/(b\,w(z))\exp[-i\vartheta(z)]$, $w(z) = w_0\sqrt{1 + z^2/z_R^2}$, $\vartheta(z) = \pm\arctan(z/z_R)$, $v_1 = ik(x^2 + y^2)/(2q(z))$, $\sigma = (kw_0)^{-1}$, $q_\pm(z) = \pm z - iz_R$. The CVF-Cosh-Gaussian beam is generated by a group of sources of strength $S_{\rm cs}$ situated at $r = 0$ and $z = z_{\rm cs}$. Proper choice of $S_{\rm cs}$ and $z_{\rm cs}$ yields the desired beam. The wave function $E(x, y, z)$ satisfies the inhomogeneous Helmholtz equation

$$\begin{aligned}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\Big)E(x, y, z) \;=\; & -S_{cs}[\delta(x - iw_0/(2b))\delta(y \pm w_0/(2b)) \\ & +\delta(x - iw_0/(2b))\delta(y \mp w_0/(2b))]\delta(z - z_{\rm cs}),\end{aligned} \tag{2}$$

Applying the Fourier transform pairs,

$$E(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{E}(f_x,f_y,z)\exp(iu_1)df_xf_y, \tag{3}$$

$$\tilde{E}(f_x,f_y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}E(x,y,z)\exp(-iu_1)dxdy, \tag{4}$$

where f_x and f_y are the spatial frequencies in the x and y directions, $u_1 = f_x x + f_y y$. From Eq. (2), $\tilde{E}(f_x,f_y,z)$ is obtained and substituted into Eq. (3), one can find that,

$$\begin{aligned} E(x,y,z) &= \frac{i}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}df_xdf_yS_{\mathrm{cs}}/\zeta[\exp(-i(if_xw_0/(2b)\mp f_yw_0/(2b)) \\ &\quad +\exp(i(if_xw_0/(2b)\mp f_yw_0/(2b))]\exp[i\zeta(z-z_{\mathrm{cs}})]\exp(iu_1), \end{aligned} \tag{5}$$

where $\mathrm{Re}(z-z_{\mathrm{cs}})>0$, $\zeta=(k^2-(f_x^2+f_y^2))^{1/2}$. Expanding ζ into series and keeping the first and second terms, we obtain $\zeta = k(1-(f_x^2+f_y^2)/(2k^2))$. Under paraxial approximation, i.e., $k^2 \gg (f_x^2+f_y^2)$, we replace ζ of the exponential part in Eq. (5) with the approximation and the other terms with k, Eq. (5) simplifies to

$$\begin{aligned} E_p(x,y,z) &= \frac{iS_{\mathrm{cs}}}{4\pi k}\exp[ik(z-z_{\mathrm{cs}})]\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}df_xdf_y[\exp(-i(if_xw_0/(2b)\mp f_yw_0/(2b)) \\ &\quad +\exp(i(if_xw_0/(2b)\mp f_yw_0/(2b))]\exp[-i(f_x^2+f_y^2)(z-z_{\mathrm{cs}})/(2k)]\exp(iu_1). \end{aligned} \tag{6}$$

The additional subscript p in $E_p(x,y,z)$ means the paraxial approximation. By using the relation

$$\int_{-\infty}^{+\infty}\exp(-ax^2+bx)dx = \sqrt{\frac{\pi}{a}}\exp\left(-\frac{b^2}{4a}\right), \tag{7}$$

one can obtain the integral of Eq. (6),

$$E_p(x,y,z) = \frac{S_{\mathrm{cs}}}{2(z-z_{\mathrm{cs}})}\exp[ik(z-z_{\mathrm{cs}})]\exp\left[\frac{ikr^2}{2(z-z_{\mathrm{cs}})}\right]\cosh\left[\frac{kw_0(x\pm iy)}{2b(z-z_{\mathrm{cs}})}\right]. \tag{8}$$

In order to obtain the CVF-Cosh-Gaussian beam for $z>0$, we suppose the input paraxial approximation distribution is given by Eq. (1) for $z=0$. By comparing Eq. (8) and Eq. (1) at $z=0$, the parameters S_{cs} and z_{cs} can be given as

$$z_{\mathrm{cs}} = ikw_0^2/2 = iz_R, S_{\mathrm{cs}} = 2iz_R\exp(-kz_R). \tag{9}$$

Substituting Eq. (9), into (5), one can obtain the exact integral expression for $E(x,y,z)$

$$\begin{aligned} E(x,y,z) &= -\frac{z_R}{2\pi}\exp(-kz_R)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}df_xdf_y 1/\zeta[\exp(-i(if_xw_0/(2b)\mp f_yw_0/(2b)) \\ &\quad +\exp(i(if_xw_0/(2b)\mp f_yw_0/(2b))]\exp[i\zeta(z-iz_{\mathrm{R}})]\exp(iu_1). \end{aligned} \tag{10}$$

From Eq. (9), the source in Eq. (2) lies external to $z>0$, and the solution given by Eq. (10) is the exact solution to the homogeneous Helmholtz equation.

3. EXACT NONPARAXIAL PROPAGATION OF A COMPLEX VARIABLE FUNCTION COSH-GAUSSIAN BEAM

Making use of the Green-function approach, we can obtain the differential representation of the CVF-Cosh-Gaussian waves. The solution of the differential equation

$$\left(\nabla_r^2+\frac{\partial^2}{\partial z^2}+k^2\right)G(r,z) = -S_{\mathrm{cs}}/(2\pi)\left(\frac{\delta(r_1)}{r_1}+\frac{\delta(r_2)}{r_2})\delta(z-z_{\mathrm{cs}}\right), \tag{11}$$

is given by [16–18] $G(r,z) = S_{\rm cs}\exp(ikR)/(4\pi R)$, where $r_1 = \sqrt{\begin{array}{c}(x-iw_0/(2b))^2+(y\pm w_0/(2b))^2\\ +(z-iz_{\rm R})^2\end{array}}$, $r_2 = \sqrt{(x+iw_0/(2b))^2+(y\mp w_0/(2b))^2+(z-iz_{\rm R})^2}$. $R=[x^2+y^2+(z-iz_R)^2]^{1/2}$. Comparing Eq. (11) and Eq. (2) and applying Eq. (9), the differential or multipole representation of the CVF-Cosh-Gaussian waves is found to be

$$E(x,y,z) = \frac{iz_R}{2}\exp(-kz_R)\left[\frac{\exp(ikr_1)}{r_1}+\frac{\exp(ikr_2)}{r_2}\right]. \tag{12}$$

From Eq. (12), we see that CVF-Cosh-Gaussian modes can be generated by superposing two complex-source-point spherical wave $S_{\rm cs}\exp(ikR)/(4\pi R)$ corresponding to the paraxial Gaussian beam.

By using the power expansion $1/\zeta$ and $\exp[i\zeta(z-z_{\rm cs})]$ in Eq. (5) and keeping the product of both series terms up to order $\sigma^{2j}(\sigma = 1/(kw_0))$, we can obtain the jth-order corrections

$$\begin{aligned} E^{(0)}(x,y,z) &= E_1^{(0)}(x,y,z)+E_2^{(0)}(x,y,z) = -\frac{i}{4}Q\exp(ikz)\\ &\quad\times\exp[iQ(x^2+y^2)/w_0^2]\cosh\left[w_0Q(x\pm iy)/b\right], \qquad (13)\\ E^{(2j)}(x,y,z) &= (-1)^j\left[(Q\rho_1)^{2j}L_j^j(-iQ\rho_1^2)E_1^{(0)}(x,y,z)\right.\\ &\quad\left.+(Q\rho_2)^{2j}L_j^j(-iQ\rho_2^2)E_2^{(0)}(x,y,z)\right], \qquad (14)\end{aligned}$$

where $E^{(0)}(x,y,z)$ is the paraxial CVF-Cosh-Gaussian beam, $E^{(2j)}(x,y,z)$ is denoting the jth-order correction to the CVF-Cosh-Gaussian beam, $Q=(z/z_R-i)^{-1}$, $E_1^{(0)}(x,y,z) = -\frac{i}{4}Q\exp(iQ\rho_1^2)$, $E_2^{(0)}(x,y,z) = -\frac{i}{4}Q\exp(iQ\rho_2^2)$, $\rho_1 = \sqrt{(x-iw_0/(2b))^2+(y\pm w_0/(2b))^2}/w_0$, $\rho_2 = \sqrt{\begin{array}{c}(x+iw_0/(2b))^2\\ +(y\mp w_0/(2b))^2\end{array}}/w_0$, and $L_j^j(\cdot)$ is the associated Laguerre polynomial of order j and degree j. It is easy to find that Eq. (13) is the same as Eq. (1). Eq. (14) is the nonparaxial expression of the CVF-Cosh-Gaussian beams for the jth nonparaxial corrections. We can find that the nonparaxial solution approaches the exact solution Eq. (12) with the increase of the parameter j. In particular, when j becomes infinite, the nonparaxial solution becomes the exact solution.

4. CONCLUSION

In conclusion, a group of complex sources that generate the CVF-Cosh-Gaussian waves has been presented. We have obtained the integral and the differential representation for the CVF-Cosh-Gaussian waves. From the integral representation of the CVF-Cosh-Gaussian waves, we derive the first three orders of nonparaxial corrections for the corresponding paraxial CVF-Cosh-Gaussian beams analytically and numerically. The angular velocity of the paraxial and nonparaxial CVF-Cosh-Gaussian beams decreases with z increasing.

ACKNOWLEDGMENT

This research was supported by the National Natural Science Foundation of China (11374108, 11374107, 10904041), the Foundation for the Author of Guangdong Province Excellent Doctoral Dissertation (SYBZZXM201227), and the Foundation of Cultivating Outstanding Young Scholars ("Thousand, Hundred, Ten" Program) of Guangdong Province in China.

REFERENCES

1. Abramochkin, E. G. and V. G. Volostnikov, *Opt. Commun.*, Vol. 102, 336, 1993.
2. Abramochkin, E. G. and V. G. Volostnikov, *Opt. Commun.*, Vol. 125, 302, 1996.
3. Abramochkin, E., N. Losevsky, and V. Volostnikov, *Opt. Commun.*, Vol. 141, 59, 1997.
4. Berry, M., *Proc. SPIE*, Vol. 3487, 6, 1998.
5. Skryabin, D. V., J. M. McSloy, and W. J. Firth, *Phys. Rev. E*, Vol. 66, 055602(R), 2002.
6. Afanasiev, K. N., S. P. Kotova, A. M. Mayorova, and V. G. Volostnikov, *SPIE*, Vol. 6054, 60540U-1, 2005.
7. Izdebskaya, Y., V. Shvedov, and A. Volyar, *J. Opt. Soc. Am. A*, Vol. 2, 171, 2008.

8. Bekshaev, A. Y. and M. S. Soskin, *J. Opt. A: Pure Appl. Opt.*, Vol. 6, S170, 2004.
9. Bekshaev, A. Y., M. S. Soskin, and M. V. Vasnetsov, *Opt. Lett.*, Vol. 31, 694, 2006.
10. Bekshaev, A. Y. and M. S. Soskin, *Opt. Lett.*, Vol. 31, 2199, 2006.
11. Deng, D. M., Q. Guo, and W. Hu, *Phys. Rev. A*, Vol. 79, 023803, 2009.
12. Deng, D. M., Q. Guo, and W. Hu, *Opt. Lett.*, Vol. 34, 43–45, 2009.
13. Deschamps, G. A., *Electron. Lett.*, Vol. 7, 684–685, 1971.
14. Felsen, L. B., *J. Opt. Soc. Am.*, Vol. 66, 751–760, 1976.
15. Shin, S. Y. and L. B. Felsen, *J. Opt. Soc. Am.*, Vol. 67, 699–700, 1977.
16. Seshadri, S. R., *Opt. Lett.* Vol. 27, 998, 2002.
17. Seshadri, S. R., *Opt. Lett.*, Vol. 27, 1872, 2002.
18. Seshadri, S. R., *Opt. Lett.*, Vol. 28, 595 2003.
19. Bandres, M. A. and J. C. Gutiérrez-Vega, *Opt. Lett.*, Vol. 29, 2213–2215, 2004.
20. Zhang, Y. C., Y. J. Song, Z. R. Chen, J. H. Ji, and Z. X. Shi, *Opt. Lett.*, Vol. 32, 292–294, 2007.
21. Deng, D. M. and Q. Guo, *Opt. Lett.*, Vol. 33, 1225, 2008.
22. Deng, D. M. and Q. Guo, *J. Opt. Soc. Am. B*, Vol. 26, 2044–2049, 2009.
23. Deng, D. M., C. D. Chen, X. Zhao, B. Chen, X. Peng, and Y. S. Zheng, *Opt. Lett.* Vol. 39, 2703–2706, 2014.
24. Yan, S. H., B. L. Yao, M. Lei, D. Dan, Y. L. Yang, and P. Gao, *Opt. Lett.*, Vol. 37, 4774, 2012.