Homogenization of Arrays of Nanorods
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Abstract—The perfectly conducting infinite wire structure has been shown to exhibit negative effective permittivity (incidence normal to the wires), and strong spatial dispersion (in conical incidence) for wavelengths much larger than the wire separation. In this work we make use of a two-scale renormalization technique to treat the more general, realistic and practically relevant finite conductivity finite wire structure. The effective medium obeys a nonlocal constitutive relation which is verified by 3D numerical simulations.

1. INTRODUCTION
The effective medium theory of wire structures has been an active direction of research in recent years, due to the technological simplicity of the structures, and their interesting properties, particularly their effective negative permittivity, which has found use in the construction of negative index metamaterials [1–3]. However, the main thrust of investigations to date have focused on infinite wire media, and perfectly conducting wires. In this work we go further, and offer a rigorous treatment of the more realistic scenario of finite wires with finite conductivity.

The structure under study is a square biperiodic array of thin wires, of length \( L \), radius \( r \) and conductivity \( \sigma \). We note the period \( d \) and the wavelength \( \lambda \). The renormalization (depicted in Fig. 1) involves a limiting process whereby the three quantities: \( r, d \) and \( 1/\sigma \) tend simultaneously to zero. The parameter governing the limiting process is noted \( \eta = d \), the period. The asymptotics of the other two parameters, \( \sigma \) and \( r \), with respect to \( \eta \) are described by fixed parameters \( \kappa \) and \( \gamma \) according to the following relations:

\[
\kappa = \frac{\pi r^2 \sigma}{\varepsilon_0 \omega \eta^2}, \quad \frac{1}{\gamma} = \eta^2 \log \left( \frac{r_\eta}{\eta} \right)
\]

(1)

where \( \omega \) is the angular frequency of the electromagnetic field. In other words the conductivity is renormalized inversely to the fill factor \( \theta_\eta = \frac{\pi r^2}{\eta^2} \), while the radius is renormalized such that the expression \( \eta^2 \log(\frac{r_\eta}{\eta}) \) remains constant.

![Figure 1: The bed-of-nails structure and the renormalization process. The conducting fibers occupy a region \( \Omega \subseteq \mathbb{R}^2 \), are oriented in the \( z \) direction, and the structure is periodic in the \( xy \) plane. Two renormalized structures are shown, corresponding to \( \eta_1 \) and \( \eta_2 \) respectively, with \( \eta_1 > \eta_2 \), \( d_{\eta_1} > d_{\eta_2} \), \( \sigma_{\eta_1} < \sigma_{\eta_2} \) and \( r_{\eta_1}/d_{\eta_1} > r_{\eta_2}/d_{\eta_2} \) (see Eq. (1)). The real physical structure corresponds by definition to \( \eta = 1 \): \( d_{\eta=1} = d \). The length \( L \) and the wavelength \( \lambda \) remain fixed, i.e., we are homogenizing in the \( xy \) plane only.](image_url)
While these expressions may at first seem obscure, they have simple intuitive interpretations. The first requires the current density to remain constant during the renormalization. The second expression requires the average internal capacitance of the wires to remain constant during renormalization. This feature is known to be essential for their asymptotic behavior (see, for instance Ref. [4]).

The question to be answered now becomes: What happens in the limit $\eta \to 0$? The answer is that the fields converge (in a precise sense described in Ref. [5]) to the unique solution of the following system:

$$
\begin{align*}
\nabla \times E &= i\omega\mu_0 H \\
\nabla \times H &= -i\omega\varepsilon_0 \left( E + \frac{P_z}{\varepsilon_0} \right) \\
\frac{\partial P_z}{\partial z} + \left( k_0^2 + \frac{2\pi\gamma}{\kappa} \right) P_z &= -2\pi\gamma\varepsilon_0 E_z, \quad z \in [-L/2, L/2] \\
\frac{\partial P_z}{\partial z} &= 0, \quad z \in \{-L/2, L/2\}
\end{align*}
$$

(2)

where $P$ is the polarization field.

As a quick check of the reasonableness of this limit system, we consider the limit of infinite conductivity and infinitely long wires. It can be seen immediately that the material is non-local, even in the long-wavelength regime. By doing a Fourier transform on the third equation of system (2) (with $\kappa \to \infty$) we obtain:

$$
\hat{P}_z = -\frac{2\pi\gamma\varepsilon_0}{(k_0^2 - k_z^2)} \hat{E}_z
$$

which gives $\varepsilon = 1 + 2\pi\gamma/(k_0^2 - k_z^2)$. This is fully consistent with existing results [6].

2. NUMERICAL RESULTS

We now proceed to test the homogeneous model by comparing it with 3D full vector simulations of the structure, i.e., we must compare the reflection, transmission and absorption coefficients and the current distribution of the homogeneous problem with those of the original bed-of-nails metamaterial. The solution to the homogeneous problem is obtained by integrating system (2) as described in Căbus et al. (to be published).

The 3D full vector simulations of the bed-of-nails metamaterial were done using the Comsol Multiphysics finite element method software package. The periodicity was implemented using Floquet-Bloch conditions [7] in the two periodic directions ($x$ and $y$), and absorbing Perfectly Matched Layers [8] in the positive and negative $z$ directions. The linearity of the materials in the structure was used to treat the incident field as a localised source within the obstacle, as detailed in Ref. [9].

Figures 2 and 3 show the results of calculations for a structure of Toray T300® carbon fibers [10] with a conductivity of: $\sigma = 5.89 \cdot 10^4 \text{(Ωm)}^{-1}$ and a radius of 3.5 microns. The wires have an aspect

![Energy efficiencies](image)

Figure 2: Transmission (solid), reflection (dot-dashed) and absorption (dashed) efficiency curves comparing the finite element solution (dot markers) and the effective medium (no markers) as a function of angle of incidence. The wire conductivity is that of Toray T300® carbon fibers $\sigma = 5.89 \cdot 10^4 \text{(Ωm)}^{-1}$ and a radius of 3.5 microns. The wires have an aspect
ratio \( L/r = 2.28 \times 10^5 \), and they were modeled using an effective impedance technique [11] which eliminates the need to mesh the interior of the wires. A finite element model in which the interior of the wires is meshed is a problem with around 3 million degrees of freedom, requiring several tens of Gigabytes of available RAM to solve. By comparison, using the impedance technique, the model of Fig. 2 (curves with markers), is a problem of approx. 62 thousand degrees of freedom, which requires less than one Gigabyte of available RAM and can therefore be solved on any sufficiently recent desktop or laptop computer.

Figures 2 and 3 illustrate the behavior which is typical of the model. The agreement remains good up to high incidence angles, and over a large wavelength domain (Fig. 5). The structure

![Image](image-url)

Figure 3: Square of the current density for the effective medium model (dashed) and the finite element solution (solid) as a function of position within the slab (which is positioned in \( z \in (0, L) \)). The structure is the same as in Fig. 2, illuminated at an angle of incidence \( \theta = 40 \) from the top. Note that the surface areas under the two curves are the same because they are proportional to the Joule dissipation rates, which are seen to be equal from Fig. 2 at the given angle of incidence.

![Image](image-url)

Figure 4: Transmission (solid), reflection (dotted-dashed) and absorption (dashed) efficiency curves comparing the finite element solution (dot markers) and the effective medium (no markers) as a function of angle of incidence. The structure has a conductivity \( \sigma = 1000 \, (\Omega m)^{-1} \), period \( d = 0.01 \, m \), and dimensionless parameters \( L/d = 50 \), \( \lambda/d = 8 \), \( r/d = 0.002 \), and \( \delta/d = 13 \). The reflection remains low for angles of incidence of up to \( 80^\circ \) even as the Joule absorption reaches almost 100\% for \( \theta > 60 \). Energy conservation is indicated by the \( \times \) markers.

![Image](image-url)

Figure 5: Transmission (solid), reflection (dotted-dashed) and absorption (dashed) efficiency curves comparing the finite element solution (dot markers) and the effective medium (no markers) as a function of wavelength. Energy conservation for the finite element model is labeled with \( \times \) markers. The structure has a conductivity \( \sigma = 3000 \, (\Omega m)^{-1} \) (in the semiconductor domain), period \( d = 0.01 \, m \), and dimensionless parameters \( L/d = 60 \), \( r/d = 0.003 \), and the angle of incidence is \( \theta = 70 \). \( \delta/r \) runs approximately from 4 to 25 from left to right over the domain of the plot. The model fails around \( \lambda \lesssim 0.1 \, m = 10d \).
is transparent in normal incidence. For increasingly oblique angles of incidence the absorption increases more or less gradually, depending on the thickness \( L \). The reflection is generally low, though it increases when approaching grazing incidence. The low reflection may be explained by the small radii of the wires: Their extremities have low capacitance, hence they exhibit very little charge accumulation, leading to an almost continuous normal component of the electric field. Certain configurations exhibit very low reflection for almost all angles of incidence, see Figs. 4 and 5 around \( \lambda = 1.2 \) m. The current density decreases roughly exponentially within the structure due to absorption.

3. DOMAIN OF VALIDITY

The boundaries of the domain of validity of the model are given by four dimensionless parameters: The ratio of the skin depth to the radius in the wires \( \delta/r \), the ratio of the wire length to the period \( L/d \), the ratio of the wavelength to the period \( \lambda/d \) and the ratio of the wire radius to the period \( r/d \).

The skin depth must be larger than the radius, due to the fact that the impedance used in defining \( \kappa \) is the static impedance which differs from the quasistatic value by an imaginary inductive term \( i\omega\mu/8\pi \) (see, for instance, Ref. [12]). Requiring this term to be negligible is equivalent to requiring that \( \delta^2/r^2 \gg 1 \). Moreover, in the rescaling process the skin-depth/radius ratio is given by

\[
\frac{\delta_\eta}{r_\eta} = \frac{\lambda}{\eta} \sqrt{\frac{1}{2\pi\kappa}}.
\]

Since \( \eta \) approaches zero in the rescaling process, it is natural to expect the homogeneous model to be valid when the skindepth is large compared to the radius.

In addition, recall that the definition of \( \gamma \) in Eq. (1) fixes the capacitance of the wires to the value for thin, long wires. Consequently, we expect the model to hold for large \( L/d \) and for small \( r/d \). To these, we must add the general requirement for all effective medium models: the wavelength must be large compared to the period.

Due to the large (four dimensional) parameter space, an exhaustive numerical exploration of the bed-of-nails structure is not feasible in a reasonable timeframe. Still, our study has made it possible to broadly determine the boundaries of the domain of applicability of the effective medium model. Roughly, one must have \( \lambda/d \gtrsim 7 - 12, \delta/r \gtrsim 4 - 8, L/d \gtrsim 20 - 30, r/d \lesssim 10 \). Our (a fortiori limited) numerical exploration of the parameter space suggests that the skindepth-to-radius ratio is often the main limiting factor, particularly when considering highly conducting wires.

4. CONCLUSION

We have tested numerically the effective medium theory of the bed-of-nails structure, whose rigorous mathematical foundation is described in Ref. [5]. We have found good agreement between the transmission, reflection and absorption efficiencies between the effective medium model and a 3D finite element model, for a broad range of angles of incidence and wavelengths. The current density in the real structure corresponds to the polarization current density of the effective medium model. The medium is nonlocal, meaning that the polarization field depends on the electric field over a region of finite size. This nonlocal behavior also means that the permittivity depends on the wavevector, so it can no longer be seen, strictly, as a property of the medium, but rather, as a property of a given wave propagating in the structure [13, 14].

The bed-of-nails structure is a medium exhibiting high absorption with low reflection. It requires a very low filling fraction of conducting material, but exhibits near perfect absorption over a wide range of angles of incidence, for sufficiently large thicknesses. The low filling fraction is useful because it allows the engineer to fill the space between the wires with materials satisfying other design constraints, such as mass density, or mechanical, chemical or thermal robustness.

REFERENCES

10. Toray Carbon Fibers America Inc.