On the \(A, B, C\) Numbers and Their Application in the Theory of Circular Waveguide with Azimuthally Magnetized Ferrite

Mariana Nikolova Georgieva-Grosse\(^1\) and Georgi Nikolov Georgiev\(^2\)

\(^1\)Meterstrasse 4, D-70839 Gerlingen, Germany  
\(^2\)Faculty of Mathematics and Informatics  
University of Veliko Tarnovo “St. St. Cyril and Methodius”, BG-5000 Veliko Tarnovo, Bulgaria

Abstract—An iterative method is worked out and applied to calculate the \(A, B, C\) numbers — positive real numbers, connected with the normalized differential phase shift, produced by the circular waveguide, containing azimuthally magnetized ferrite which sustains normal \(TE_{01}\) mode. It is based on the recently formulated definition of the quantities in question and uses the positive purely imaginary zeros of a certain complex Kummer confluent hypergeometric function, determined for a varying imaginary part of its first complex parameter. The results of investigation are presented graphically. The influence of the normalized guide radius and of the magnitude of off-diagonal ferrite permeability tensor element on the numbers is analyzed. A numerical example is presented, demonstrating the way of figuring out the phase shift with their help.

1. INTRODUCTION

The \(A, B, C\) numbers have been advanced in the theory of azimuthally magnetized circular ferrite waveguides, propagating normal \(TE_{01}\) modes, built through the medium of the complex confluent hypergeometric functions, as a means for computation of the differential phase shift provided by these structures \([1, 2]\). Finding the quantity mentioned is important in view of the possible application of geometries in the development of nonreciprocal digital phase shifters for microwave frequencies \([1–9]\). This requires knowledge of the numbers, the information for which still is scarce \([1, 2]\).

The discussion is confined here to the circular waveguide, entirely filled with ferrite that supports normal \(TE_{01}\) mode. An iterative technique is elaborated that might yield \(A, B, C\) in the whole area of phase shifter operation of configuration \([3, 8]\). For the purpose, the definitions: \(A = A_− − A_+\), \(B = B_− − B_+\) and \(C = C_− − C_+\) of the numbers, introduced newly \([2]\), are harnessed in which \(A_±\), \(B_±\) and \(C_±\) are set through the roots of a specially constructed biquadratic equation, involving the positive purely imaginary zeros of wave function for propagation — the complex Kummer function \([10]\). The effect of the magnitude of the off-diagonal ferrite permeability tensor element \(|α|\) and of the normalized guide radius \(r_0\) on the quantities studied, is examined. The benefit of the latter in the exact reckoning of the differential phase shift, called into being by the set-up, is shown.

2. ITERATIVE METHOD FOR COMPUTATION OF THE \(A, B, C\) NUMBERS

The pair \(|α|, r_0\) of positive real numbers \(|α| < 1\) and \(r_0 > ν_{1,n}\), where \(ν_{1,n}\) is the \(n\)th zero of Bessel function \(J_1(z/2)\), \(n = 1, 2, 3, \ldots, z > 0\) is chosen, subject to the requirement \([3]\):

\[
ν_{1,n} < r_0\sqrt{1 − α^2} < L(c, n)/|α|
\]

(1)

in that the symbols \(L(c, n)\) and \(c\) in it designate certain real positive and integer numbers, resp., e.g., \(L(3, 1) = 6.5936541068\) \([3]\). Next, the \(n\)th positive purely imaginary roots \(ζ^{(c)}_{k,n}\) in \(x_0\) (in \(z_0\)) of the equation \([1–4, 8]\):

\[
Φ(a, c; x_0) = 0,
\]

(2)

written in terms of the complex Kummer function \([10]\) with \(a = c − 2 − jk\), \(c = 3\), \(x_0 = jz_0\), \(k\), \(z_0\) — real, \(-∞ < k < +∞\), \(z_0 > 0\), are found for an arbitrarily selected numerical equivalent of the parameter \(k_±\), satisfying the condition \([2]\):

\[
1 − \overline{β}_2^2 ≥ 4\overline{β}_2|k_±|
\]

(3)

in which \([2–4]\)

\[
\overline{β}_2 = ζ_{k_±,n}^{(c)}/(2r_0).
\]

(4)
Afterwards the roots
\[ \sigma^2_{1\pm,2\pm} = 0.5 \left[ (1 - \beta^2_{2\pm}) \pm \sqrt{(1 - \beta^2_{2\pm})^2 - 4 \times 4 \beta^2_{2\pm} k^2_{2\pm}} \right] \]  
(5)
of the equation
\[ \sigma^4_{\pm} - (1 - \beta^2_{2\pm}) \sigma^2_{\pm} + 4 \beta^2_{2\pm} k^2_{2\pm} = 0 \]  
(6)
are specified \[2\]. The parameter \( k_{2\pm} \) is varied, until \( \sigma_{1\pm} \) (or \( \sigma_{2\pm} \)) coincides with the singled out \( |\alpha| \) within the framework of the prescribed accuracy. Then, the values of \( \sigma_{1\pm} \) and \( \sigma_{2\pm} \) obtained, are substituted in the expressions \[2\]:

\[ A_{\pm} = \sigma_{1\pm}/\sigma_{2\pm} \quad \text{or} \quad A_{\pm} = \sigma_{2\pm}/\sigma_{1\pm}, \]  
(7)

\[ B_{\pm} = \sigma_{1\pm}\bar{r}_0 \quad \text{or} \quad B_{\pm} = \sigma_{2\pm}\bar{r}_0, \]  
(8)

\[ C_{\pm} = (\sigma_{1\pm}/\sigma_{2\pm})\bar{r}_0 \quad \text{or} \quad C_{\pm} = (\sigma_{2\pm}/\sigma_{1\pm})\bar{r}_0, \]  
(9)

Figure 1: \( A \) numbers for normal \( TE_{01} \) mode vs.: (a) \( |\alpha| \) in the interval \( 0 \div 1.0 \) with \( \bar{r}_0 \) as parameter; (b) \( \bar{r}_0 \) in the interval \( 3.5 \div 10 \) with \( |\alpha| \) as parameter; (c) \( \bar{r}_0 \) in the interval \( 3.5 \div 7 \) with \( |\alpha| \) as parameter; (d) \( \bar{r}_0 \) in the interval \( 3.8 \div 4.6 \) with \( |\alpha| \) as parameter.
resulting in the $A_{\pm}$, $B_{\pm}$ and $C_{\pm}$ numbers. If $\sigma_{1\pm} \equiv |\alpha|$ ($\sigma_{2\pm} \equiv |\beta|$), the second (first) group of
formulae is used. Finally, the quantities of interest are counted from [2]:

$$A = A_{-} - A_{+},$$
$$B = B_{-} - B_{+},$$
$$C = C_{-} - C_{+}.$$  

(The subscripts “+” and “-” answer to positive and negative sign of $k$, resp.)

3. BASIC CHARACTERISTICS OF THE $A$, $B$, $C$ NUMBERS

Figures 1–3 illustrate the dependence of $A$, $B$ and $C$ on $|\alpha|$ and $\bar{r}_0$ with solid lines, assuming $n = 1$. In Figs. 1(a), 2(a) and 3(a), (b), $|\alpha|$ is considered as a variable and $\bar{r}_0$ — as a parameter, whereas in all the rest — the opposite holds. The ends (origins) of the curves in the first (second) set of Figures are denoted by circles with abscissas, determined through the formula: $|\alpha_{cr}| = \sqrt{1 - (v_{1,n}/\bar{r}_0)^2}$ ($\bar{r}_{0cr} = v_{1,n}/\sqrt{1 - |\alpha|^2}$) [3]. $LEn_{1}$ — ($LEn_{1}$ —) dashed envelope lines link them. (The letter, preceding the notation given, indicates the number with which the curves in question are associated.) In each of the sets the values of the components of the pairs $\{|\alpha_{cr}|, \bar{r}_0\}$ or $\{|\alpha|, \bar{r}_{0cr}\}$, describing the relevant circles, are the same for the three numbers. The parameter $\bar{r}_0$ ($|\alpha|$) relates to the whole line, depicting certain number in a Figure from the first (second) group, including its end (starting) point, while the answering to it $|\alpha_{cr}|$ ($\bar{r}_{0cr}$) — to the latter only (cf. Fig. 1(a), resp. Figs. 2(b) and 3(d)). To avoid overcrowding in Figs. 2(a) and 3(a), (b) the numerical equivalents of $\bar{r}_0$ solely, are written at the termination of the curves with which they are connected. For the same reason in Figs. 1(b), (c), (d) and 3(c), the value of relevant parameter $|\alpha|$ is shown at the beginning of lines only, whereas in Figs. 2(b) and 3(d), that of $\bar{r}_{0cr}$ is also presented. All characteristics in Figs. 2(a) and 3(a), (b), have a common initial point at $|\alpha| \to 0$, labeled by $\bar{r}_{0in} = 3.832$. The value of the abscissa $\bar{r}_{0cr} = 3.832$ for $|\alpha| \to 0$ of the origin of the $LEn_{1}$ — lines in the second series of Figures (cf. Figs. 2(b) and 3(d)) coincides with the one of $\bar{r}_{0in}$. Obviously, $|\alpha_{cr}|$ ($\bar{r}_{0cr}$) is the largest (smallest) value of the parameter $|\alpha|$ ($\bar{r}_0$) at which the $A$, $B$ and $C$ numbers exist for certain $\bar{r}_0$ ($|\alpha|$); $\bar{r}_{0in}$ is the smallest value that $\bar{r}_{0cr}$ might attain when $|\alpha| \to 0$. The graphs reveal that the $A$ numbers almost do not alter with $|\alpha|$ and follow approximately the function $1/|\alpha|$, the $B$ ones increase practically linearly with $|\alpha|$ and are not affected by the change of $\bar{r}_0$ and the impact of both parameters on the $C$ numbers, is slight.

4. APPLICATION

Equation (2) governs the propagation and Eq. (4) yields the eigenvalue spectrum of the normal $TE_{01}$ modes with phase constant $\beta_{\pm}$ and radial wavenumber $\beta_{2\pm} = \sqrt{\omega^2 \varepsilon_0 \mu_0 \varepsilon_r (1 - \alpha_{\pm}^2)} - \beta_{1\pm}^2$ in

![Figure 2](image-url)

Figure 2: $B$ numbers for normal $TE_{01}$ mode vs.: (a) $|\alpha|$ in the interval $(0 \div 1.0)$ with $\bar{r}_0$ as parameter; (b) $\bar{r}_0$ in the interval $(3.5 \div 10)$ with $|\alpha|$ as parameter.
the circular waveguide of radius \( r_0 \), entirely filled with azimuthally magnetized ferrite, described by a Polder permeability tensor of off-diagonal element \( \alpha = \gamma M_r / \omega \) (\( \gamma \) — gyromagnetic ratio, \( M_r \) — ferrite remanent magnetization, \( \omega \) — angular frequency of the wave) and a scalar permittivity \( \varepsilon = \varepsilon_0 \varepsilon_r \), if it is fulfilled

\[
k_\pm = \frac{\alpha_\pm \beta_\pm}{(2 \beta_2 \pm)} \quad \text{and} \quad z_0 = 2 \beta_2 \bar{r}_0 \left( \frac{\beta}{\beta_0 \sqrt{\varepsilon_r}} \right),
\]

\( \bar{r}_0 = \beta_0 r_0 \sqrt{\varepsilon_r} \), and \( \beta_0 = \omega \sqrt{\varepsilon_0 \mu_0} \). Moreover, \( \bar{r}_{0\text{cr}} (|\alpha_{cr}|) \) is the critical value of the normalized (barred) guide radius (of the off-diagonal element) at which the transmission ceases both for positive \( (\alpha_+ > 0, k_+ > 0) \) and negative \( (\alpha_- < 0, k_- < 0) \) magnetization [3]; \( \bar{r}_{0\text{in}} \) stands for \( \bar{r}_{0\text{cr}} \) when the load is dielectric (|\alpha| = 0) of permittivity \( \varepsilon_r \). Thus, the circles (the envelopes), designating the ends (origins) of the \( A \), \( B \), \( C \) curves, are relevant to the cutoff state of configuration. They mark the limit of the domain of existence of the numbers from the side of lower frequencies. The sequence of inequalities (1) represents the condition for phase shifter operation of this structure, i.e., it specifies the set of values of parameters \(|\alpha|, \bar{r}_0\) for which it might provide differential phase shift \( \Delta \beta = \beta_- - \beta_+ \). The latter could be calculated in normalized form by means of the formula [2]:

\[
\Delta \beta = \frac{A B}{C}.
\]
Numerical example: The case $\bar{r}_0 = 4$ and $|\alpha| = 0.1$ is considered. Employing for it the method, worked out in Section 2, yields: $A = 0.513\,613$, $B = 0.205\,445$ and $C = 2.054\,453$. Accordingly, it is obtained: $\Delta \bar{\beta} = 0.051\,361$. Since finding the quantities $A$, $B$ and $C$ for each set of parameters by the reiterative scheme is difficult, it is suggested to use for this purpose the Figures presented. This allows to get $\Delta \bar{\beta}$ from Eq. (13) for all $\bar{r}_0 \in (3.832 \div 10)$ and $|\alpha| \in (0 \div 1.0)$ in which graphical outcomes for the numbers studied are available. (All results relate to normal $TE_{01}$ mode $(n = 1)$.)

5. CONCLUSION

Based on the definitions of the positive real numbers $A$, $B$ and $C$ in terms of the roots of a certain biquadratic equation, an iterative technique is developed and harnessed to specify their values. The equation mentioned is equivalent to the ones, binding the normalized in a suitable way phase constant $\bar{\beta}$ of the normal $TE_{0m}$ modes in the azimuthally magnetized ferrite-loaded circular waveguide, resp. the magnitude of the off-diagonal ferrite permeability tensor element $|\alpha|$ with the normalized radial wavenumber $\bar{\beta}_2$ (with the eigenvalue spectrum of waves), expressed through the zeros of a definite complex Kummer function and the normalized guide radius $\bar{r}_0$. The analysis of the graphically shown upshots indicates that the $A$ $(B)$ numbers are almost independent of $|\alpha| (\bar{r}_0)$ and that the $C$ ones slightly vary when both parameters change. The application of quantities examined in the counting up the differential phase shift, provided by the structure pointed out, is manifested.

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