Excitation and Propagation of Whistler Waves in a Magnetoplasma Containing Density and Magnetic-field Nonuniformities

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Abstract—Excitation and propagation of whistler-mode waves in a magnetoplasma containing cylindrically symmetric nonuniformities of plasma density or external static magnetic field are studied. Using a rigorous solution for the total source-excited field comprising both the discrete and continuous parts of the spatial spectrum of waves, the radiation resistance of a loop antenna in the presence of such structures is determined. Conditions are found under which the radiation resistance of the loop antenna located in a weakly nonuniform plasma-density depletion or magnetic-field enhancement can be notably greater than that in a homogeneous magnetoplasma whose parameters coincide with those near the nonuniformity axis.

1. INTRODUCTION
Whistler-mode waves guided by cylindrical density enhancements in a magnetoplasma have received much careful study and there are many accounts of them (see, e.g., [1] and references therein). However, there exists very little theory of the phenomena related to the source-excited whistler waves in the guiding structures located in an unbounded background magnetoplasma and formed by density depletions or nonuniformities of an external static magnetic field. Earlier studies of the whistler wave guidance by such plasma structures employed various approximate approaches such as the geometrical optics or the WKB approximation [2, 3]. In the present work, the full-wave approach is used to analyze the features of excitation and propagation of whistler waves in a magnetoplasma containing the above-mentioned cylindrical structures in the case where their radii are comparable to or less than typical wavelengths of the guided modes.

2. FORMULATION OF THE PROBLEM
We consider a cold unbounded collisionless magnetoplasma containing a cylindrical nonuniformity in which either the plasma density or external static magnetic field is a function of radial distance from the nonuniformity axis. This axis is taken as the $z$ axis of a cylindrical coordinate system $(\rho, \phi, z)$. Parallel to this axis is an external static magnetic field $\vec{B}_0 = B_0 \hat{z}_0$. The plasma is assumed to be described by the cold-plasma dielectric tensor $\varepsilon$ whose nonzero elements are written as follows: $\varepsilon_{\rho\rho} = \varepsilon_{\phi\phi} = \epsilon_0 \varepsilon$, $\varepsilon_{\rho\phi} = -\varepsilon_{\phi\rho} = -i \epsilon_0 g$, and $\varepsilon_{zz} = \epsilon_0 \eta$, where $\epsilon_0$ is the permittivity of free space. General expressions for the quantities $\varepsilon$, $g$, and $\eta$ are determined by the medium parameters and are given elsewhere [1].

It is assumed that the frequency $\omega$ belongs to the whistler range

$$\omega_{\text{LH}} \ll \omega < \omega_{\text{H}} \ll \omega_{\text{p}},$$

where $\omega_{\text{LH}}$ is the lower hybrid frequency, $\omega_{\text{H}}$ is the electron gyrofrequency, and $\omega_{\text{p}}$ is the electron plasma frequency.

We take the following models of cylindrical nonuniformities in a magnetoplasma. The first one is a density duct in a uniform static magnetic field. In this case, the plasma density $N$ is a constant, $\tilde{N}$, in the inner core $\rho < a_0$, a constant, $N_a$, in the outer region $\rho > a_1$, and varies smoothly from $\tilde{N}$ to $N_a$ within the duct wall $a_0 < \rho < a_1$ by the law

$$N(\rho) = \frac{\tilde{N} + N_a + (\tilde{N} - N_a) \sin \pi (\rho - a)/ (a_1 - a_0)]}{2},$$

where $a = (a_0 + a_1)/2$. The second kind of nonuniformity to be considered is a magnetic-flux tube in a plasma of constant density. In this case, the external magnetic field $B_0$ takes constant values $\tilde{B}_0$ and $B_a$ in the regions $\rho < a_0$ and $\rho > a_1$, respectively, and varies in the interval $a_0 < \rho < a_1$...
The electromagnetic field is excited by a ring electric current whose density (with the \( \exp(i \omega t) \) time dependence dropped) is given by
\[
\vec{J}(\rho, z) = \hat{\delta} q I_0 \delta(\rho - b) \delta(z),
\]
which corresponds to a circular loop antenna commonly used in appropriate experiments. Here, \( \delta \) is the Dirac function, \( b \) is the loop radius, and \( I_0 \) is the total antenna current.

Our main task is to study the efficiency of excitation of guided and unguided waves which are radiated from source (3) located in a density duct or magnetic-flux tube. To do this, we should obtain the solution of the Maxwell equations in the presence of a cylindrical nonuniformity in the plasma.

3. FIELD EXPANSION IN THE PRESENCE OF A CYLINDRICAL NONUNIFORMITY

Due to the azimuthal symmetry of the problem, the solution of the source-free Maxwell equations can be sought in terms of the modal fields
\[
\begin{bmatrix}
\vec{E}_{s,\alpha}(\vec{r}, q) \\
\vec{B}_{s,\alpha}(\vec{r}, q)
\end{bmatrix} = \begin{bmatrix}
\vec{E}_{s,\alpha}(\rho, q) \\
\vec{B}_{s,\alpha}(\rho, q)
\end{bmatrix} \exp(-i k_0 p_{s,\alpha}(q) z),
\]
where \( q \) is the transverse wave number in the ambient magnetoplasma (\( \rho > a_1 \)), normalized to the free-space wave number \( k_0 = \omega/c \); the function \( p_{s,\alpha}(q) \) describes the dependence of \( p \), the axial wave number normalized to \( k_0 \), on the transverse wave number \( q \) for the “ordinary” (\( \alpha = o \)) and “extraordinary” (\( \alpha = e \)) characteristic waves of the ambient uniform plasma; the subscript \( s \) denotes the wave propagation direction (\( s = - \) and \( s = + \) designate waves propagating in the positive and negative directions of the \( z \) axis, respectively); and \( \vec{E}_{s,\alpha}(\rho, q) \) and \( \vec{B}_{s,\alpha}(\rho, q) \) are the vector wave functions describing the radial distribution of the field of a mode corresponding to the transverse wave number \( q \) and the indices \( s \) and \( \alpha \). The functions \( p_{s,\alpha}(q) \) obey the relation \( p_{+\alpha}(q) = p_0(q) = -p_{-\alpha}(q) \), where \( p_0(q) \) is written as
\[
p_0(q) = \left[ \varepsilon_a - \frac{1}{2} \left( 1 + \frac{\varepsilon_a}{\eta_a} \right) q^2 + \chi_a \eta_a \right]^{1/2},
\]
\[
R_\alpha(q) = \left[ \frac{1}{4} \left( 1 - \frac{\varepsilon_a}{\eta_a} \right) q^4 - \frac{g_a^2}{\eta_a q^2} + g_a^2 \right]^{1/2}. 
\]
Here, \( \varepsilon_a, g_a, \) and \( \eta_a \) denote the quantities \( \varepsilon, g, \) and \( \eta \), respectively, in the outer region, and \( \chi_a = -\chi_\varepsilon = -1 \). It is assumed that \( \text{Re} R_\alpha(q) > 0 \) and \( \text{Im} p_0(q) < 0 \). Note that the functions \( \vec{E}_{s,\alpha}(\rho, q) \) and \( \vec{B}_{s,\alpha}(\rho, q) \) can be expressed in terms of two scalar functions \( E_{\phi; s,\alpha}(\rho, q) \) and \( B_{\phi; s,\alpha}(\rho, q) \).

Following work [1], it can be shown that in the considered case, a complete set of normal modes over which the total field can be expanded comprises the discrete spectrum of transversely localized eigenmodes, which are guided by the plasma nonuniformity, and the continuous spectrum of unguided modes that are necessary to describe the radiation field.

The unguided modes correspond to positive real values of \( q \). In the uniform outer region (\( \rho > a_1 \)), the fields of the continuous-spectrum modes are written as follows:
\[
E_{\phi; s,\alpha}(\rho, q) = i \left[ \sum_{k=1}^{2} C_{s,\alpha}^{(k)}(q) H_{1}^{(k)}(k_0 q \rho) + D_{s,\alpha}(q) H_{1}^{(2)}(k_0 q \rho) \right],
\]
\[
B_{\phi; s,\alpha}(\rho, q) = -e^{-1} \left[ \sum_{k=1}^{2} C_{s,\alpha}^{(k)}(q) n_{s,\alpha}^{(1)} H_{1}^{(k)}(k_0 q \rho) + D_{s,\alpha}(q) n_{s,\alpha}^{(2)} H_{1}^{(2)}(k_0 q \rho) \right]. 
\]
Here, \( H_{1}^{(1,2)} \) are Hankel functions of the first and second kinds, \( C_{s,\alpha}^{(1,2)} \) and \( D_{s,\alpha} \) are coefficients to be determined, and
\[
n_{s,\alpha}^{(1,2)}(q) = -\varepsilon_a \left[ \left( q_{\alpha}^{(1,2)} \right)^2 + p_a^2(q) + \frac{g_a^2}{\varepsilon_a} - \varepsilon_a \right] (p_{s,\alpha}(q) g_a)^{-1},
\]
\[
q_{\alpha}^{(1)} = q, \quad q_{\alpha}^{(2)} = q_{\alpha}(q) = \left[ \varepsilon_a - \frac{p_a^2}{\varepsilon_a}(q) - \frac{g_a}{\varepsilon_a} \left( g_a - \frac{\eta_a p_{s,\alpha}(q)}{n_{s,\alpha}^{(1)}(q)} \right) \right]^{1/2},
\]
where it is assumed that \( \text{Im} q_{\alpha}(q) < 0 \). The law that can be obtained from (2) by making the replacements \( \tilde{N} \to \tilde{B}_0, N_0 \to B_0, \) and \( N(\rho) \to B_0(\rho) \).
In the nonuniform part of the cylindrical structure, the fields of unguided modes cannot be expressed in terms of known functions, and the field equations must be solved numerically. There are four independent solutions for the field in the region \( \rho < a_1 \), of which two solutions, hereafter denoted as \( \bar{E}_{\phi;\rho,a}(\rho,q) \), \( \bar{B}_{\phi;\rho,a}(\rho,q) \) and \( \bar{E}_{\rho;\rho,a}(\rho,q) \), \( \bar{B}_{\rho;\rho,a}(\rho,q) \), are regular at \( \rho = 0 \). Then the solution for the field in the region \( \rho < a_1 \) is written as

\[
E_{\phi;\rho,a}(\rho,q) = i \sum_{k=1}^{2} A_{\phi;\rho,a}^{(k)}(\rho,q) \bar{E}_{\phi;\rho,a}^{(k)}(\rho,q), \quad B_{\phi;\rho,a}(\rho,q) = -c^{-1} \sum_{k=1}^{2} A_{\phi;\rho,a}^{(k)}(\rho,q) \bar{B}_{\phi;\rho,a}^{(k)}(\rho,q),
\]

where \( A_{\phi;\rho,a}^{(1,2)} \) are coefficients to be determined. It is evident that in the uniform inner core \( \rho < a_0 \), the particular solutions \( \bar{E}_{\phi;\rho,a}(\rho,q) \) and \( \bar{B}_{\phi;\rho,a}(\rho,q) \) reduce to cylindrical functions (see [1] for more details). Knowing the values of these functions and their derivatives at \( \rho = a_0 \), one can easily find the functions \( \bar{E}_{\phi;\rho,a}(\rho,q) \) and \( \bar{B}_{\phi;\rho,a}(\rho,q) \) in the nonuniform region \( a_0 < \rho < a_1 \) by numerically solving the wave equations for each \( q \). Such a procedure automatically ensures the continuity of the tangential field components at \( \rho = a_0 \). Next, satisfying the continuity conditions for the tangential field components at \( \rho = a_1 \), we arrive at the system of linear equations for unknown coefficients \( A_{\phi;\rho,a}^{(1,2)}, C_{\phi;\rho,a}^{(1,2)} \), and \( D_{\phi;\rho,a} \). This system can be represented in matrix form as \( \mathbf{S} \cdot \vec{G} = C_{\phi;\rho,a}^{(1)} \vec{F} \), where the elements of the column vector \( \vec{G} \) are given by the expressions \( G_{1,2} = A_{\phi;\rho,a}^{(1,2)}, G_3 = C_{\phi;\rho,a}^{(1)}, \) and \( G_4 = D_{\phi;\rho,a} \). The elements of the matrix \( \mathbf{S} \) and the components of the column vector \( \vec{F} \), which are not written here for brevity, are expressed in an obvious manner via particular field solutions entering the representations of the tangential fields on both sides of the interface \( \rho = a_1 \). The above-mentioned matrix equation gives four linear relationships for five coefficients \( A_{\phi;\rho,a}^{(1,2)}, C_{\phi;\rho,a}^{(1,2)} \), and \( D_{\phi;\rho,a} \), so that one of these coefficients can be taken arbitrary [1]. For numerical calculations, it is most convenient to put \( C_{\phi;\rho,a}^{(1)} = \det|\mathbf{S}| \) and then determine the remaining coefficients.

The transversely localized eigenmodes (also called the discrete-spectrum modes) guided by the cylindrical nonuniformity correspond to discrete complex quantities \( q_n \) \( (n = 1, 2, \ldots) \). It turns out that these quantities are zeros of the coefficient \( C_{\phi;\rho,a}^{(1)} \) and have the negative imaginary part (i.e., \( \text{Im} q_n < 0 \)). The substitution of \( q_n \) into \( p_{s,a}(q) \) yields the axial wave numbers \( p_{s,n} \) of the eigenmodes. It is adopted that \( p_{\pm,n} = \pm p_{n} \). For shortening the writing, we represent the fields of the discrete-spectrum modes in the form

\[
\begin{bmatrix}
\bar{E}_{s;\rho,a}(\rho,\eta) \\
\bar{B}_{s;\rho,a}(\rho,\eta)
\end{bmatrix} =
\begin{bmatrix}
\bar{E}_{s;\rho,a}^{(1)}(\rho) \\
\bar{B}_{s;\rho,a}^{(1)}(\rho)
\end{bmatrix} \exp(-ik_0 p_{s,n} \rho),
\]

where \( \bar{E}_{s;\rho,a}(\rho) = \bar{E}_{s,a}(\rho, q_n) \) and \( \bar{B}_{s;\rho,a}(\rho) = \bar{B}_{a}(\rho, q_n) \).

With allowance for the performed analysis, the field excited by an axisymmetric source in the presence of a cylindrical nonuniformity is given by the following expansion over the discrete- and continuous-spectrum modes outside the source region:

\[
\begin{bmatrix}
\bar{E}(\rho) \\
\bar{B}(\rho)
\end{bmatrix} = \sum_n a_{s,n} \begin{bmatrix}
\bar{E}_{s;\rho,a}(\rho) \\
\bar{B}_{s;\rho,a}(\rho)
\end{bmatrix} + \sum_\alpha \int_0^\infty a_{s,a} \left[ \begin{bmatrix}
\bar{E}_{s,a}(\eta) \\
\bar{B}_{s,a}(\eta)
\end{bmatrix} \right] dq,
\]

where \( a_{s,n} \) and \( a_{s,a} \) are the excitation coefficients of the respective modes. In expansion (10), one should put \( s = + \) for \( z > 0 \) and \( s = - \) for \( z < 0 \). Next, following the well-known technique developed for finding the excitation coefficients of modes of open waveguides [1], we can obtain for source (3)

\[
a_{\pm,n} = I_0 2\pi b N^{-1} E_{\phi;\rho,a}^{(T)}(b), \quad a_{\pm,n}(q) = I_0 2\pi b N^{-1}(q) E_{\phi;\rho,a}^{(T)}(b, q). \quad (11)
\]

Here, the superscript \( (T) \) denotes fields taken in a medium described by the transposed dielectric tensor \( \epsilon^T \), and the normalization quantities \( N_n \) and \( N_a(q) \) for modes are given by

\[
N_n = \frac{2\pi}{\mu_0} \int_0^\infty \left[ \bar{E}_{+,n}(\rho) \times \bar{B}_{-,n}^{(T)}(\rho) - \bar{E}_{-,n}^{(T)}(\rho) \times \bar{B}_{+,n}(\rho) \right] \cdot \hat{z}_0 \rho \, d\rho, \\
N_a(q) = - \frac{16\pi}{Z_0 k_0^2} \left( \frac{d\alpha(q)}{dq} \right)^{-1} \left[ 1 + \eta^{-1}_{\alpha} \left( \eta_{+,n}^{(1)} \right)^2 \right] C_{+,n}^{(2)}(q) C_{+,\alpha}^{(2)}(q),
\]

where \( \mu_0 \) and \( Z_0 \) are the permeability and wave impedance of free space, respectively.
The total radiation resistance of the loop antenna with current (3) is written as
\[ R_\Sigma = \frac{2P_\Sigma}{|I_0|^2} = \sum_n R_n + R_{cs}, \]  
where \( P_\Sigma \) is the total radiated power and
\[ R_n = -2\pi b \text{Re} \left( I_0^{-1} a_{s,n} E_{\phi,s,n}(b) \right), \quad R_{cs} = -2\pi b \text{Re} \int_0^\infty I_0^{-1} a_{s,e}(q) E_{\phi,s,e}(b, q) dq. \]  
The quantities \( R_n \) and \( R_{cs} \) are the partial radiation resistances corresponding to the discrete- and continuous-spectrum modes, respectively. We emphasize that the “ordinary” wave, which is evanescent in range (1), does not contribute to the radiation.

4. NUMERICAL RESULTS
An examination shows that in range (1), volume eigenmodes can exist only in ducts with decreased density or tubes with enhanced static magnetic field. The normalized axial wave numbers of such modes lie in the range \( 2\tilde{\varepsilon}_{1/2} < p < 2\varepsilon_{1/2} \). Hereafter, the tilde quantities refer to the inner region of the cylindrical nonuniformity. In addition, such structures can guide no more than one eigenmode of surface type with the axial wave number \( p < 2\varepsilon_{1/2} \). However, the surface modes are of little interest since they are excited inefficiently by source (3). As an example, Figs. 1 and 2 show the field components of the lower-order volume eigenmode (guided mode whose axial wave number \( p \) is minimum) with the azimuthal index \( m = 0 \) for a duct with decreased plasma density and a tube with enhanced static magnetic field, respectively. Note that the values of dimensionless parameters chosen for numerical calculations are appropriate to conditions typical of the corresponding modeling laboratory experiments (see [1, 4] and references therein). Similarity of the field distributions in the figures is indicative of the fact that the features of eigenmodes on the guiding structures of both types are almost identical.

Figure 3 shows the results of numerical computations of the total radiation resistance \( R_\Sigma \) of the loop antenna with current (3) as a function of the antenna radius \( b \) in the presence of a density depletion duct and the analogous dependences of the partial radiation resistances \( R_n \) for the volume eigenmodes. For comparison, Fig. 3(a) also shows the total radiation resistance of the same antenna immersed in a homogeneous magnetoplasma whose density coincides with that in the inner region of the duct. We can infer from Fig. 3 that for the chosen parameters, the radiation resistance of the loop antenna located in a weakly nonuniform density depletion can be several times greater than the radiation resistance of the same source immersed in a homogeneous plasma with the corresponding density. Note that in the considered case, the surface eigenmode and the continuous-spectrum modes give the very small contribution to \( R_\Sigma \). It turns out that the contribution of the continuous-spectrum modes to the total radiation resistance is negligible if \( \tilde{\omega}_p b/c > 1 \). Thus, the maxima of \( R_\Sigma \) in our case are determined by the maxima of \( R_n \) for the dominant volume modes. With increasing duct radius \( a \), the number of guided modes accordingly increases, the maxima of \( R_\Sigma \) become smoothed, and the gain in the total radiation resistance no longer takes place.

We do not present the numerical results for the radiation resistance of the loop antenna in a tube with enhanced static magnetic field for the parameters used in Fig. 2 since the dependences of \( R_\Sigma \) and \( R_n \) on \( b \) in this case turn out to be similar to those presented in Fig. 3.

Figure 1: Field components of the lower-order axisymmetric volume eigenmode as functions of \( \rho \) for a density depletion duct if \( \omega_\rho/\omega_H = 0.3, \omega_{\phi}/\omega_H = 0.82, \omega_H/\omega_\rho = 29.3, \omega_H a/c = 0.42, a_0/a = 0.8, \) and \( a_1/a = 1.2 \). The quantities \( \omega_\rho \) and \( \omega_H \) refer to the plasma of the outer region.
Figure 2: Field components of the lower-order axisymmetric volume eigenmode as functions of $\rho$ for a tube with enhanced static magnetic field if $\omega/\omega_H = 0.24$, $\omega_0/\omega_H = 0.82$, $\omega_p/\omega_H = 29.3$, $\omega_H a/c = 0.42$, $a_0/a = 0.8$, and $a_1/a = 1.2$. The same notations as in Fig. 1.

Figure 3: (a) Total radiation resistance of the loop antenna as a function of its radius $b$ in the cases where the antenna is located in a density depletion duct with the above-described density profile (curve 1) and in a homogeneous magnetoplasma whose density coincides with that in the inner region of the duct (curve 2). (b) Partial radiation resistances for the eigenmodes as functions of the loop radius $b$ (the curves are labeled in order of increasing axial wave numbers of the eigenmodes). The same values of parameters as in Fig. 1.

5. CONCLUSION

The features of excitation and propagation of whistler-mode waves in a magnetoplasma containing weakly nonuniform guiding structures in the form of either a duct with decreased density or tube with enhanced static magnetic field have been found to be similar in the whistler frequency range. It has been established that under certain conditions, the radiation resistance of the loop antenna in such plasma structures can be notably greater than that in a homogeneous magnetoplasma whose parameters coincide with those near the nonuniformity axis.

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