An Efficient 3D Integral Equation Method for Computation of Electromagnetic Wavefields in a Layered Configuration Containing Inhomogeneous Objects

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Abstract—This paper is concerned with the source-type of integral equation to compute the electromagnetic scattering by an inhomogeneous 3D object in a planar layered medium in the frequency domain. By decomposing the scattered field into a particular and a general constituent, the structure of the integral operator of the integral equation is constructed. The particular constituent represents the scattered field inside the layer that embodies the contrasting object, due to the presence of virtual contrast sources inside the inhomogeneous object, while the general constituent represents the interaction with the other layers due to the presence of source distributions on each side of the layer that embodies the contrasting object. The particular constituent has a convolution structure in all spatial directions. The general constituent consists of two terms; one has again a convolution structure with respect to all spatial coordinates, while the other has a convolution structure with respect to the horizontal coordinates and a correlation structure in the vertical coordinates. These properties facilitate a fast and efficient computation of the integral operator with the help of Fast Fourier Transforms. In view of numerical efficiency, it is desirable to keep the spatial derivatives outside the Fourier integral, rather than to consider them as spectral multiplications with the wave vector inside the Fourier integral. The method is applied to simulate the geophysical low-frequency electromagnetic problem, i.e., the controlled-source electromagnetic (CSEM) method.

1. INTRODUCTION

We consider the electromagnetic scattering by an inhomogeneous 3D object in a planar layered medium in the frequency domain. The inhomogeneous object, with a bounded supporting domain \( D \), is located in the heterogeneous layer \( n = 0 \) of a horizontally layered medium (see Fig. 1). The vectors \( x = (x_1, x_2, x_3) \) and \( x’ = (x_1’, x_2’, x_3’) \) denote the vectorial position in \( \mathbb{R}^3 \). We use the complex time factor \( \exp(-i\omega t) \). In each homogeneous layer, \( n = -M, \ldots, 0, \ldots, N \), the medium is characterized by a constant conductivity \( \sigma_n \). We assume that the frequency is low enough to neglect dielectric displacement currents. The permeability \( \mu \) is constant in whole space. Then the wave number is given by \( k_n = \omega(\sigma_n\mu)^{\frac{1}{2}} \). We define the incident wavefield, \( \{ E^{\text{inc}}, H^{\text{inc}} \} \), as the field in the layered background medium in absence of the inhomogeneous object in \( D \), and the scattered field in the layered background as \( \{ E^{\text{sct}}, H^{\text{sct}} \} = \{ E, H \} - \{ E^{\text{inc}}, H^{\text{inc}} \} \). For an object in a homogeneous embedding, the scattering problem at hand can be formulated as a domain integral equation over the object domain \( D \). By a weak formulation, in which the spatial derivatives are kept outside the domain integral, a very efficient computational method is arrived at [1]. In this paper we extend this weak formulation to a horizontally layered embedding.

2. CONSTRUCTION OF THE PARTICULAR AND GENERAL SOLUTIONS

The scattered electromagnetic field in the layered configuration (including the object) is written as a superposition of a particular solution and a general solution, viz.,

\[
\{ E^{\text{sct}}, H^{\text{sct}} \} = \{ E^{\text{prt}}, H^{\text{prt}} \} + \{ E^{\text{gen}}, H^{\text{gen}} \}.
\]

We assume that the particular solution vanishes outside the layer \( x_{3,0} \leq x_3 \leq x_{3,1} \), while inside this layer it satisfies Maxwell’s equations. Inside this layer the particular solution is given by the following contrast source integral representation

\[
E^{\text{prt}}(x) = (k_0^2 + \nabla \nabla \cdot) \int_D G^{(0)}(x - x’) \chi(x’) E(x’) dx’,
\]
where the contrast is given by $\chi(x') = \sigma(x')/\sigma_0 - 1$. Note that $\chi E$ can be considered as the contrast source that generates the scattered wavefield. With the notation $x = x_T + x_3 i_3$ and $x_T = \{x_1, x_2\}$, the Green function may be written as a Fourier Bessel integral,

$$G^{(0)}(x - x') = \frac{\exp(ik_0|x - x'|)}{4\pi|x - x'|} = \frac{i}{4\pi} \int_0^\infty \frac{\kappa}{\gamma_0} J_0(\kappa |x_T - x_3'|) \exp(i\gamma_0|x_3 - x_3'|) d\kappa,$$

where $\gamma_0 = (k_0^2 - \kappa^2)^{\frac{1}{2}}$. The Green function in this domain integral representation for the particular solution has a convolution structure both in the horizontal direction and in the vertical direction.

![Figure 1: The object domain D in a homogeneous layer of a horizontally layered medium.](image)

Let us assume that $x_{3,\text{max}}$ and $x_{3,\text{min}}$ are the maximum and minimum extensions of the object $D$ in the $x_3$-direction (see Fig. 1). We observe that for $x_3 > x_{3,\text{max}}$ the particular wave field consists of a down-going plane wave spectrum and for $x_3 < x_{3,\text{min}}$ it consists of an up-going wave spectrum. The particular constituent of our field solution does satisfy the Maxwell equations and satisfies all the interface conditions, except at $x_3 = x_{3,0}$ and $x_3 = x_{3,1}$, where the particular solution jumps. Hence, at these levels extra surface sources occur, both of the electric type and of the magnetic type. These surface sources generate the general constituent of our field solution, being a superposition of a field with zero vertical electric field (TE-wavefields) and a field with zero vertical magnetic field (TM-wavefields). After decomposition of the particular wavefield in a TE-part and a TM-part, the general wavefield in up- and down-going wavefields in each layer, and application of the appropriate continuity conditions at the interfaces between the layers, (see [2, 3]), the general wavefield constituent is obtained as $E^{\text{gen}} = E^{\text{gen},(1)} + E^{\text{gen},(2)}$, where

$$E^{\text{gen},(1)}(x) = k_0^2 \int_D G^{(1)}_{\text{TE}}(x_T - x_T', x_3 + x_3') \chi(x') E_T(x') dx'$$

$$+ k_0^2 \int_D G^{(1)}_{\text{TM}}(x_T - x_T', x_3 + x_3') \chi(x') E_3(x') dx'$$

$$- (\nabla \nabla \cdot) \int_D G^{(1)}_{\text{TM}}(x_T - x_T', x_3 + x_3') \chi(x') E(x') dx'$$

$$+ (\nabla_T \nabla_T \cdot) \int_D G^{(1)}_{\text{EM}}(x_T - x_T', x_3 + x_3') \chi(x') E_T(x') dx',$$

(4)
and
\[
E_{\text{gen}, (2)}(x) = k_0^2 \int_{D} G_{\text{TE}}^{(2)}(x_T - x_T', x_3 - x_3') \chi(x') E_T(x') dx' \\
+ k_0^2 \int_{D} G_{\text{TM}}^{(2)}(x_T - x_T', x_3 - x_3') \chi(x') E_3(x') dx' \\
+ (\nabla \nabla \cdot) \int_{D} G_{\text{EM}}^{(2)}(x_T - x_T', x_3 - x_3') \chi(x') E_T(x') dx',
\]

(5)
in which we have made the decomposition of the electric field vector \( E \) and the differential operator \( \nabla \) in a transverse part and a vertical part, viz., \( E = E_T + E_3 \) and \( \nabla = \nabla_T + i\kappa \partial_3 \). The overbar above the nabla operator indicates that we have to change the sign of the vertical derivative, i.e., \( \nabla = (\partial_1, \partial_2, -\partial_3) \). This is directly related to the fact that the general solution \( E_{\text{gen}, (1)} \) corresponds to wavefields being generated by secondary sources at the interfaces at \( x_3 = x_3,0 \) and \( x_3 = x_3,1 \) and reflected an odd number of times by these latter interfaces. The Green functions in the domain integral representations for this part of the general solution have a convolution structure in the horizontal directions and a correlation structure in the vertical direction. The general solution \( E_{\text{gen}, (2)} \) correspond to wavefields being generated by secondary sources at the interfaces at \( x_3 = x_3,0 \) and \( x_3 = x_3,1 \) and reflected an even number of times by these latter interfaces. The Green functions in the domain integral representations for this part of the general solution have a convolution structure both in the horizontal directions and in the vertical direction. Further the various Green functions are given as Fourier Bessel integrals:

\[
G^{(1)}_{\text{TE}}(x_T, x_3) = \frac{i}{4\pi} \int_{0}^{\infty} \frac{J_0(\kappa|x_T|)}{70} \left\{ R_{\text{TE}}^{(0,-1)} \exp[i\gamma_0(x_3 - 2x_3,0)] + R_{\text{TE}}^{(0,1)} \exp[i\gamma_0(2x_3,1 - x_3)] \right\} d\kappa,
\]

\[
G^{(1)}_{\text{TM}}(x_T, x_3) = \frac{i}{4\pi} \int_{0}^{\infty} \frac{J_0(\kappa|x_T|)}{70} \left\{ R_{\text{TM}}^{(0,-1)} \exp[i\gamma_0(x_3 - 2x_3,0)] + R_{\text{TM}}^{(0,1)} \exp[i\gamma_0(2x_3,1 - x_3)] \right\} d\kappa,
\]

\[
G^{(1)}_{\text{EM}}(x_T, x_3) = \frac{i}{4\pi} \int_{0}^{\infty} \frac{J_0(\kappa|x_T|)}{70} \left\{ R_{\text{EM}}^{(0,-1)} \exp[i\gamma_0(x_3 - 2x_3,0)] + R_{\text{EM}}^{(0,1)} \exp[i\gamma_0(2x_3,1 - x_3)] \right\} d\kappa,
\]

(6)

where

\[
R^{(0,-1)}_{\text{EM}} = k_0^2 R^{(0,-1)}_{\text{TE}} + R^{(0,-1)}_{\text{TM}} \quad \quad R^{(0,1)}_{\text{EM}} = k_0^2 R^{(0,1)}_{\text{TE}} + R^{(0,1)}_{\text{TM}},
\]

(7)

and

\[
R^{(0,-1)}_{\text{TE}} = \frac{\tilde{R}_{\text{TE}}^{(0,-1)}}{1 - R_{0,-1}^{\text{TE}} R_{0,1}^{\text{TE}} \exp(2i\gamma_0d_0)}, \quad R^{(0,-1)}_{\text{TM}} = \frac{\tilde{R}_{\text{TM}}^{(0,-1)}}{1 - R_{0,-1}^{\text{TM}} R_{0,1}^{\text{TM}} \exp(2i\gamma_0d_0)},
\]

\[
R^{(0,1)}_{\text{TE}} = \frac{\tilde{R}_{\text{TE}}^{(0,1)}}{1 - R_{0,1}^{\text{TE}} R_{0,1}^{\text{TE}} \exp(2i\gamma_0d_0)}, \quad R^{(0,1)}_{\text{TM}} = \frac{\tilde{R}_{\text{TM}}^{(0,1)}}{1 - R_{0,1}^{\text{TM}} R_{0,1}^{\text{TM}} \exp(2i\gamma_0d_0)},
\]

\[
R^{(0,-1|0,1)}_{\text{TE}} = \frac{\tilde{R}_{\text{TE},0,-1}^{(0,-1)}}{1 - R_{0,-1}^{\text{TE}} R_{0,1}^{\text{TE}} \exp(2i\gamma_0d_0)}, \quad R^{(0,-1|0,1)}_{\text{TM}} = \frac{\tilde{R}_{\text{TM},0,-1}^{(0,-1)}}{1 - R_{0,-1}^{\text{TM}} R_{0,1}^{\text{TM}} \exp(2i\gamma_0d_0)}.
\]

(8)
The reflection coefficients $\tilde{R}_{0,-1}^{\text{TE}}$ and $\tilde{R}_{0,-1}^{\text{TM}}$ denote the generalized reflection coefficients of the stack of layers above the interface at $x_3 = x_{3,0}$, relating the amplitude of the down-going TE- and TM-wavefields to the up-going wavefields in the region above the object in layer $n = 0$. Similarly, the reflection coefficients $\tilde{R}_{0,1}^{\text{TE}}$ and $\tilde{R}_{0,1}^{\text{TM}}$ denote the generalized reflection coefficients of the stack of layers below the interface at $x_3 = x_{3,1}$, relating the amplitude of the up-going TE- and TM-wavefields to the down-going wavefields in the region below the object in layer $n = 0$. These coefficients can easily be computed by a recursive scheme [4].

An integral equation for the total electric field is obtained by observing that

$$E(x) = E^{\text{inc}}(x) + E^{\text{prt}}(x) + E^{\text{gen},(1)}(x) + E^{\text{gen},(2)}(x), \quad x \in \mathbb{D}. \tag{9}$$

Substituting Eqs. (2), (4) and (5) in the above equation we arrive at

$$E(x) = E^{\text{inc}}(x) + (\nabla \nabla_{T}^{-}) \int_{\mathbb{D}} \left[ G^{(0)} - G^{(1)}_{\text{TM}} + G^{(2)}_{\text{TM}} \right] (x, x') \chi(x')(x') E_{T} dx'$$

$$+ (k_{0}^{2} + \nabla \theta_{3}) \int_{\mathbb{D}} \left[ G^{(0)} + G^{(1)}_{\text{TM}} + G^{(2)}_{\text{TM}} \right] (x, x') \chi(x') E_{3}(x') dx'$$

$$+ k_{0}^{2} \int_{\mathbb{D}} \left[ G^{(0)} + G^{(1)}_{\text{TE}} + G^{(2)}_{\text{TE}} \right] (x, x') \chi(x') E_{T}(x') dx'$$

$$+ (\nabla_{T} \nabla_{T}^{-}) \int_{\mathbb{D}} \left[ G^{(1)}_{\text{EM}} + G^{(2)}_{\text{EM}} \right] (x, x') \chi(x') E_{T}(x') dx' \tag{10}$$

which shows that, once the spatial 3D Fourier transforms of the Green functions have been computed, the computation of the electric field in the object involves three forward 3D Fourier transforms and seven inverse 3D Fourier transforms.

In view of numerical efficiency, it is desirable to keep the spatial derivatives outside the domain integrals, rather than to consider them as spectral multiplications with the wave vector inside the Fourier Bessel integrals. The discretization of the integral equation at hand runs along similar lines as in [1]. The spatial derivatives are replaced by a finite difference rule. The convolution and correlation properties of the Green function facilitate a fast and efficient computation of the integral operator with the help of Fast Fourier Transforms. Hence, the total computational time of this approach will be proportional to $20 N_{\text{iter}} N \log(N)$, where $N_{\text{iter}}$ is the total number of iterations of a BiCGSTAB type of solver and $N$ is the number of discretization points of the 3D rectangular domain that encloses the object domain $\mathbb{D}$. The Fourier Bessel integrals in the Green functions are truncated and computed with a trapezoidal rule. Furthermore, since the integral operator acts as a filter for the product of the fields and the contrast, the number of Fourier components in the Green functions, that are needed to accurately calculate the integral operator, can be reduced to a number that is related to the sampling theorem.

Figure 2: Configuration in the $(x_{1}, x_{3})$-plane. The source at depth of 200 m operates at 0.25 Hz. The length of the object in the $x_{2}$-direction is 1 km. The receivers are at $x_{2} = 0, x_{3} = 300$ m, with a total extent of 20 km in the $x_{1}$-direction.
3. NUMERICAL EXAMPLE

Our domain integral method is used for simulating the geophysical low-frequency electromagnetic problem, i.e., the controlled-source electromagnetic (CSEM) method in marine geophysics. The 3D object \( D \) of interest is a rectangular domain of 3 km by 0.1 km by 1 km, in which an oil-based medium with an electrical conductivity of 0.01 S/m is present (see Fig. 2). It is embedded in a four-layer configuration. The sea surface is located at \( x_3 = 0 \). Above the sea surface we have a semi-infinite air layer with a conductivity of 0.1 mS/m. Below the sea surface we have a water layer with a conductivity of 3 S/m. In this water layer we assume that we have a vertical electric dipole at position \((0, 0, 0.2 \text{ km})\). We have 21 receivers distributed equidistantly along the the receiver line at \(-10 \text{ km} \leq x_1 \leq 10 \text{ km}, x_2 = 0 \) and \( x_3 = 0.4 \text{ km} \). The object is embedded in a layer with conductivity of 1 S/m. Below this layer a semi-infinite layer is present with conductivity of 0.5 S/m. The frequency of operation is 0.25 Hz. The object domain is discretized in a grid of 120 by 40 by 4. The grid size is 25 m by 25 m by 25 m. The BiCGSTAB iterative solver reaches the error criterion of \( 10^{-3} \) in 13 iterations. The results for the vertical electric field at the receiver locations are presented as solid blue lines in Fig. 3. For comparison we also plot the results by using an integral-equation based preconditioned finite-difference method [5]. In the latter method, we deal with a grid of 400 by 40 by 120 and a grid size of 50 m by 50 m by 25 m. The results are plotted as dashed red lines and we observed that the agreement is excellent. The computational time of the domain integral equation method is about 148 s on a PC with 3.04 GHz processor. It is noted that the computer time of the present method can be further reduced when we store the Bessel functions in the Fourier Bessel integrals for various arguments in the computer memory. Presently, these Bessel functions are repeatedly computed for the same arguments.

![Figure 3: The logarithm of the amplitude (left) and the phase (right) of the vertical electric field component \( \hat{E}_3 \) at the receiver locations, generated by a vertical electric dipole in the \( x_3 \)-direction: incident field (top), scattered field (bottom). Solid blue lines: integral equation method. Dashed red lines: finite difference method.](image-url)
4. CONCLUSIONS
We have developed an efficient method to compute the scattering by an object in a planar layered medium. The method extend the weak form of the source-type domain integral equation of the scattering problem of an object in a homogeneous embedding to a similar one for a layered embedding. Specifically the low-frequency (diffusive) electromagnetic problem has been dealt with. For an example representative for the geophysical CSEM configuration the efficiency and the accuracy has been shown.

REFERENCES