A Method to Reduce the Oscillations of the Solution of Time Domain Integral Equation Using Laguerre Polynomials

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Abstract—In this paper, the reason of oscillations appeared in late time low energy region of time domain electric filed integral equation using laguerre polynomials is analyzed. And then, we propose an approach to reduce the oscillations by combining frequency and time domain. According to the Fourier transform relation between frequency responses and time responses, the solution of TDIE method with Laguerre expansion is absolutely convergent. Replace the spectrum of oscillation parts of time domain data with accurate frequency data by MOM. The stable time domain data can be obtained by performing IFFT to the revised frequency data. Numerical results are presented to illustrate the efficiency of this approach.

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1. INTRODUCTION

For solving the time domain integral equation, the marching on in time (MOT) method [1] is usually employed. However, as pointed by many researchers, some of the TD-EFIE formulations associated with the MOT method suffers from late-time instability, which usually takes the form of an exponentially increasing high frequency oscillation. In order to eliminate the instability of the MOT methods, the approximate averaging method [2] and the implicit scheme based on backward finite difference [3] approximation or central finite difference [4] methodology are proposed. Even though employing these methods, the solution obtained by using MOT has still exponentially increased late-time oscillations but only extend the span of the stable region. Young-seek Chung and Sarkar [5–7] proposed a marching on order method of using the expansion by weighted Laguerre polynomials for temporal coefficients of Hertz vector. The advantages of using the weighted Laguerre polynomials are the solution is independent of the time discretization, and all the computations are carried out using spatial variables. So the exponentially increasing solution will not emerge in late time. But there still exist some oscillations in late-time low energy region due to oscillations characteristic of weighted Laguerre polynomials. Oscillation phenomenon effects on obtaining other electromagnetic characteristic from the late time scattering data. In this paper, the reason of oscillations appeared in late time low energy region of time domain electric filed integral equation using laguerre polynomials is analyzed, and then, we consider absolute convergence of the time solution and the Fourier transform relation between temporal response and frequency response. An approach based on discrete Fourier transforms to improve the stabilities in late time is proposed.

2. THE REASON OF OSCILLATIONS IN LATE-TIME

It is assumed that we have obtained the coefficients for the Hertz vector \( P_{n,i} \), \( i = 1, 2 \cdots u \). Details of the method can be found in [6]. We can derive the scattered electric field as:

\[
E^s(r, t) = \sum_{v=0}^{u} a_v \varphi_v(\hat{t})
\]

where \( \hat{t} = st - \frac{r - \hat{r} \cdot \hat{r}}{c} \), \( \varphi_v(t, s) = e^{-s^2 t^2} L_v(s \cdot t) \), \( a_v = \eta s^2 \sum_{n=1}^{N} 0.25 P_{n,v} \sum_{i=0}^{v-1} (v-i) P_{n,i} \int f_n(r) \times \hat{r} ds \times \hat{r} \), \( s \) is time scale factor, \( \hat{t} \) is the scaled delay time. \( L_v(t) \) is Laguerre polynomial for order \( v \), \( \varphi_v(t) \) is weighted Laguerre polynomial. \( f_n(r) \) is the RWG basis function [8], \( \eta \) is the wave impedance in the medium surrounding the scatter.

So the last result of the electric field can be presented by the sum of Laguerre polynomials with different order. We know that when the signals with instantaneous jump pass through band-limited system, or the signals are expanded with Fourier series of finite orders, Gibbs phenomena...
will emerge. In the same way, oscillations will occur in late time low energy domain while the scattered electric fields are expanded by Laguerre polynomials with finite orders. The only difference is the frequency in truncation place. It is a frequency span in the truncation place with Laguerre expansion, unlike Fourier expansion where the frequency is a point. The weighted Laguerre polynomials are plotted in Fig. 1.

\[ E^s(f) = \int_{-\infty}^{\infty} E^s(r, t) \cdot e^{-j\omega t} dt = \sum_{v=0}^{\infty} a_v \varphi_v(\tilde{t}) \cdot e^{-j\omega t} dt = \sum_{v=0}^{\infty} a_v \int_{-\infty}^{\infty} \varphi_v(\tilde{t}) \cdot e^{-j\omega t} dt \]

From formula (5), we can see that the spectrum of temporal electric field also can be spanned by spectrum of the weighted Laguerre polynomials with different orders. The spectrums of the weighted Laguerre function of different order, are plotted in Fig. 2, are oscillatory too. So there should exist an oscillation region in the frequency spectrum transformed from time data.

It is accordant for time response and frequency response through the Fourier transform, and we can obtain frequency response of arbitrary frequency by using MOM [8,9] method. If we replace the oscillation region with right frequency data by MOM method, and then make Inverse-Fourier transform for entire spectrum, a stable time data can be obtained.

Because the incident fields forms are different in frequency domain and time domain, we need to compensate frequency data by MOM for amplitude and phase to keep consistency in the data of Fourier transform of time domain data.

The computational cost is proportional to the number of the frequency point which need replacing. If the oscillation region of spectrum is wide, the computational time will increase. In order to reduce computational cost, we only need to calculate partial frequency data from MOM, and then smooth it through interpolation/extrapolation of frequency domain data.

3.1. Amplitude and Phase Supplement
It is assumed that the scattering system of the target is a linear system. And its pulse response is \( h(t) \), and frequency response is \( H(f) \). In time domain, we use incident Gaussian plane wave

\[ E^{in1}(r, t) = E_0 \frac{1}{T\sqrt{\pi}} e^{-[1/T(\text{ct}-\text{ct}_0-\tilde{r} \cdot \hat{k})]^2}; \]

where \( \hat{k} \) is the unit vector in the propagation direction of the incident wave, \( T \) is pulse width, \( c \) is the velocity of light in the air, \( \tilde{r} \) is a position vector relative to the origin, and \( t_0 \) represents a time
delay of the peak from the origin. In frequency domain, we use incident plane wave
\[ E^{\text{in}2}(r, f) = \vec{E}_0 e^{-j\vec{k} \cdot \vec{r}} \]  
where \( \vec{k} \) is the propagation vector, \( \vec{r} \) is a position vector relative to the origin.

After excited by incident field, the scattering electric fields are denoted as \( E^{s1}(t) \) in time domain and \( E^{s2}(f) \) in frequency domain. According to the relation between time response and frequency response, plotted in Fig. 3, we have
\[ E^{s1}(t) = E^{\text{in}1}(t) \otimes h(t) \]  
\[ E^{s2}(f) = E^{\text{in}2}(f) \cdot H(f) \]  
The Fourier transform of temporal scattering field is
\[ E^{s1}(f) = E^{\text{in}1}(f) \cdot H(f) \]  
Do Fourier transform for the incident Gaussian plane wave, by Fourier transform formula \( e^{-\alpha x^2} \rightarrow \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{4\alpha}} \) and vector formula \( \vec{k} = \frac{\omega}{c} \vec{k} \), and assume that \( x = t - t_0 - \frac{\vec{r} \cdot \vec{k}}{c} \).

The Fourier transform of the incident Gaussian plane wave can be written as
\[ E^{\text{in}1}(r, f) = \int_0^\infty E^i(r, t) e^{-j\omega t} dt \]  
\[ = \vec{E}_0 \left( \frac{1}{T \sqrt{\pi}} \right) e^{-\frac{\omega^2 (t_0 + \vec{r} \cdot \vec{k}/c)^2}{2}} \int_{t_0 + \vec{r} \cdot \vec{k}/c}^{t_0 + \vec{r} \cdot \vec{k}/c} e^{-c^2/T^2 x^2} e^{-j\omega x} dx \]  
\[ = \vec{E}_0 \left( \frac{1}{\sqrt{2\pi c}} \right) e^{-T^2 \omega^2/4c^2} e^{-j\omega t_0} e^{-j\vec{k} \cdot \vec{r}} \]  
Inserting (4) into (6), and inserting (8) into (7), and compare them, we have
\[ E^{s1}(f) = \left( \frac{1}{\sqrt{2\pi c}} \right) e^{-T^2 \omega^2/4c^2} e^{-j\omega t_0} E^{s2}(f) \]  
Therefore, if we make a compensation with amplitude \( \left( \frac{1}{\sqrt{2\pi c}} \right) e^{-T^2 \omega^2/4c^2} \) and phase \( e^{-j\omega t_0} \) for frequency scattering electric field by MOM, then we can replace the spectrum of temporal scattering electric field with the frequency data from MOM.

3.2. Interpolation/Extrapolation of Frequency Domain Data
To maximize the use of the given information and to smooth the data in the frequency domain, we use a method based on the Hilbert transform as described in [10, 11]. The method is an iterative technique to extrapolate/interpolate frequency domain data relying on fact that the underlying time-domain data is causal, the real and imaginary parts of the frequency domain response have to be related through the Hilbert transform. The details of the method can be found in [10, 11].

Figure 4 compares the real and imaginary parts of the interpolated data with original data. Clearly, padded with zeros to the sample point 221 to 250, the reconstruction is quite accurate using this technique.
3.3. The Operation Procedure

Assume that we have a sequence of time domain data with length $N_t$, defined to be $x(i), i = 1 \cdots N_t$.

1) Perform a $N_f$ point FFT on this time domain data and define the resulting sequence to be $E^{s1}(k), k = 1 \cdots N_f$.

2) Find the oscillation samples $E^{s1}(N_1 : N_2)$, select samples point $[N_3 : N_4]$ from $[N_1 : N_2]$ ($N_1 < N_3 < N_4 < N_2$, and $N_2 - N_4 + N_3 - N_1 < N_t/5$). Calculate scattering field sequence $E^{s2}(N_3 : N_4)$ by using MOM, replace the $E^{s1}(N_3 : N_4)$ with $(1/\sqrt{2\pi} e^{-T^2/4\sigma^2} e^{-j\omega t_0} E^{s2}(N_3 : N_4)). \omega$ is the frequency in samples point $[N_3 : N_4]$.

3) Smooth the new $N_f$ point frequency data with Hilbert transform. Define the resulting sequence to be $E^{s\text{new}}(k), k = 1 \cdots N_f$.

4) Perform a $N_f$ point IFFT on $E^{s\text{new}}(k)$.

4. NUMERICAL EXAMPLES

In this section, two perfectly conducting structures are tested to validate the proposed method, all the structures are illuminated by the incident field with $\vec{E}_0 = \hat{x}, \vec{k} = -\hat{z}$. All time domain data are obtained from time domain electric filed integral equation using Laguerre polynomials, and frequency domain data from MOM.

For the fist example, we consider a PEC sphere with radius 0.06 m that is centered at the origin as shown in Fig. 5. The number of triangular patches is 160 and the number of unknowns is 240. In this case we use $T = 0.1/c$ s, $t_0 = 0.6/c$ s, $dt = 0.01/c$ s, and time scale factor $s = 1.0 \times 10^9$. The frequency step $\Delta f = 1.5 \times 10^7$ Hz. (The time domain data to be Fourier transformed can be formed by concatenating an appropriate number of zeros, to control the frequency step).

Figure 5(b) shows the frequency spectrum from MOM and the Fourier transform of the time domain data from Laguerre method. We can see that there exists oscillating region from sample 90 to 130. We calculate frequency data from sample 101 to 110, and reconstruct time data using the procedure described in Section 3.3. Figs. 5(a) and (c) shows the reconstructed result compared with the original time data and the accurate IFFT data of frequency domain data from MOM. We can see that the reconstructed data reduced the oscillation emerged in late time. It is almost equal to the IFFT data of frequency domain data from MOM.

The second target is a PEC plane model with length 0.2 m, front-wingspan is 0.1125 m, back-wingspan is 0.07 m, and up-wingspan is 0.03 m, as shown in Fig. 6. The number of triangular patches is 686 and the number of unknowns is 1029. In this case we use $T = 0.06/c$ s and $t_0 = 0.6/c$ s, $dt = 0.0075/c$ s, and time scale factor $s = 8.0 \times 10^9$. The frequency step $\Delta f = 1.2 \times 10^7$ Hz.

Figure 6(b) shows the frequency spectrum from MOM and the Fourier transform of the time domain data from Laguerre method. We can see that there exists oscillation region from sample 80 to 140. Replaced frequency samples 101 to 120 by MOM, and reconstructed time data using the procedure are described in Section 3.3. From Figs. 6(a) and (c), we can see that the reconstructed
data reduce the oscillation emerged in late time. It is almost equal to the IFFT data of frequency
domain data from MOM too.

Figure 5: Simulation of PEC sphere. (a) scattering field in time domain at the point (0,0,1000 m), (b) spectrum of scattering field at the point (0,0,1000 m), (c) scattering field in late time domain.

Figure 6: Simulation of PEC plane model. (a) scattering field in time domain at the point (0,0,1000 m), (b) spectrum of scattering field at the point (0,0,1000 m), (c) scattering field in late time domain.
5. CONCLUSIONS

In this paper, we have reduced the oscillation in late time domain by using TDIE with Laguerre polynomials through combining a few partial frequency domain data with entire time domain data. Two examples have demonstrated the good reconstructed result by using a few frequency data to replace the dilapidate spectrum of time domain data. It is important to obtain other electromagnetic characteristic from the late time scattering data.

REFERENCES