Green’s Function Retrieval by Crossconvolutions

Evert Slob
Department of Geotechnology, Delft University of Technology, The Netherlands

Abstract — Several formulations exist for retrieving the Green’s function from cross correlation of (passive) recordings at two locations. Usually these formulations retrieve Green’s functions from sources on a closed boundary. Then they apply to media without losses inside the domain spanned by the sources. Until recent, these formulations were only developed for wave phenomena. Now Green’s function representations for Green’s functions for electromagnetic fields in matter exist. When they exploit cross correlations, the sources must lie on a boundary of a lossless medium and outside this boundary the medium can be lossy. Then such methods can be exploited for passive applications using transient or ambient noise sources, either natural or man-made. For seabed logging methods this is not a realistic scenario because possible are necessarily located in the conductive medium. A second formulation employs cross convolutions to retrieve the Green’s function and this formulation remains valid when the media in the domain spanned by the sources is conductive. We derive here general exact electromagnetic Green’s function retrieval and simplify the obtained results for practical applications.

DOI: 10.2529/PIERS060907163222

1. INTRODUCTION
Since the early theoretical work of Clearbout [3] and Cox [4], and the experimental work of Weaver and Lobkis [8, 21], many others have contributed to our understanding of Green’s function retrieval from cross-correlating two recordings in a noise field [2, 6, 9, 11, 15, 17–19, 22]. From one-dimensional and pulse-echo experiments the subject has evolved to arbitrary three-dimensional media, ranging from having statistical properties to being fully deterministic.

Recently, based on the principle of reciprocity representations have been derived for electromagnetic waves and fields in media with non-zero conductivity values or other relaxation mechanisms, using transient or uncorrelated noise sources [13, 14]. Here we derive representations of electromagnetic Green’s functions for conductive media. When the sources lie on a closed boundary, crosscorrelation type techniques cannot be used for recordings of diffusive electromagnetic fields. An example of under which conditions a correlation type technique can be used for recordings of diffusive fields can be found in [16], who assumes that sources are distributed in a finite volume with a particular strength that is related to the local loss factor. When the sources are on a boundary and the medium inside the volume spanned by this boundary is lossless, then also cross correlation techniques can be used for exact retrieval of the full heterogeneous medium Green’s function. In a seabed logging configuration this is not realistic, because if sources would exist in the air then still almost no signal would penetrate the sea layer. Therefor here we exploit the possibility of starting from the reciprocity theorem of the time-convolution type [12]. We investigate sources located on the boundary of a finite domain. We show here that for seabed logging applications an exact Green’s function representation can be obtained, using sources on the boundary, by convolving two recordings at two different locations using the reciprocity theorem of the time-convolution type. The retrieved Green’s function corresponds to the actual heterogeneous and anisotropic medium. This requires either electric and magnetic current sources or, in case only electric sources are used, dipole and quadrupole sources. We discuss the effects of the simplifying assumption, where the quadrupole source is approximated by an equivalent dipole source, which is necessary for practical applications in a measurement situation.

2. CAUSAL FIELD EQUATIONS
All representations are valid in the time domain for transient or noise signals, but we develop our theory in the frequency domain. To this end, we define the time-Fourier transform of a space-time dependent vector-quantity as

\[ \hat{u}(x, \omega) = \int_{t=0}^{\infty} \exp(-j\omega t) u(x, t) dt, \] (1)
where \( j \) is the imaginary unit and \( \omega \) denotes angular frequency.

In the space-frequency domain Maxwell’s equations in matter are given in matrix-vector form [20] by

\[
D_x \hat{u} + \left[ B + j\omega A \right] \hat{u} = \hat{s},
\]

where the field vector \( \hat{u}^T(x, \omega) = (\hat{E}^T, \hat{H}^T) \), \( \hat{E} \) and \( \hat{H} \) being the electric and magnetic field vectors and the superscript \( T \) denotes transposition, \( \hat{s}^T(x, \omega) = -((\hat{J}^e)^T, (\hat{J}^m)^T) \) is the source vector, with \( \hat{J}^e \) and \( \hat{J}^m \) the external electric and magnetic current density vectors, while \( D_x \) is the matrix of spatial differential operators given by

\[
D_x = \left( \begin{array}{cc} O & D_0^T \\ D_0 & O \end{array} \right), \quad D_0 = \left( \begin{array}{ccc} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{array} \right).
\]

The material matrices are defined as \( A = \text{blockdiag} (\varepsilon, \mu) \), with \( \varepsilon \) and \( \mu \) the electric permittivity and magnetic permeability tensors and \( \hat{B} = \text{blockdiag} (\hat{\sigma}^e, \hat{\sigma}^m) \), with \( \sigma^e \) and \( \sigma^m \) the electric and magnetic conductivity tensors. Notice that we have defined the electric permittivity and magnetic permeability tensors and \( \varepsilon^d \) as frequency independent functions. This presents no loss of generality because all possible relaxation mechanisms are incorporated in the frequency dependent conductivity tensors. Further a real-valued diagonal matrix \( K = K^{-1} \) is introduced as \( K = \text{diag}(-1, -1, -1, 1, 1, 1) \), such that \( KD_x K = -D_x = -D_0^T \), \( KAK = A = A^T \) and \( KBK = B^T \), where the latter two definitions represent the non-negative definiteness of the material tensors. Such media are called self-adjoint or reciprocal [5].

In the next section we use the causal fields in the time-convolution type reciprocity relations. A reciprocity theorem in general interrelates two independent states, labeled \( A \) and \( B \), in one and the same domain, but the fields, sources and the medium parameters in the two states need not be the same [1, 5, 7, 10]. In our derivations here we assume all medium parameters to be the same in both states (\( A_A = A_B = A \) and \( B_A = B_B = B \)). First we establish an expression for source receiver reciprocity and then we formulate the integral representation for Green’s function retrieval. The Green’s function corresponds to the actual heterogeneous and anisotropic medium.

### 3. CONVOLUTION-TYPE ELECTROMAGNETIC GREEN’S FUNCTION REPRESENTATIONS

To allow for relaxation phenomena and non-zero electric and magnetic conduction currents we now consider the reciprocity theorem of the time-convolution type. We use the interaction quantity

\[
u^A_K D_x u_B + \nu^A_B K D_x u_B.
\]

Substituting Equation (2) for the two states in this interaction quantity, integrating the result over the domain \( \mathbb{D} \) and applying Gauss’ divergence theorem to the interaction quantity, we find the global form of the reciprocity theorem of time-convolution type as [20]

\[
\int_{\mathbb{D}} \left[ \hat{u}^T_A K \hat{s}_B - \hat{s}^T_A K \hat{u}_B \right] \, d^3 x = \oint_{\partial \mathbb{D}} \hat{u}^T_A K N_s \hat{u}_B d^2 x,
\]

where the minus sign in the left-hand side arises because use has been made of \( D_0^T K = -K D_x \). Notice that in the convolution type representations the relaxation and loss mechanisms do not occur in the expression for reciprocal media and hence we do not have to assume that the medium is lossless.

To localize the electric field receiver locations at \( x_A \) and \( x_B \) we specify the artificial point sources by replacing the space and frequency dependent \( 6 \times 1 \) vector \( \hat{s}_A \) by the \( 6 \times 6 \) matrix \( \mathbf{I} \delta (x - x_A) \), \( \mathbf{I} \) being the identity matrix. The corresponding \( 6 \times 1 \) field vector \( \hat{u}_A \) is replaced by the \( 6 \times 6 \) Green’s matrix \( \hat{G} (x, x_A, \omega) \), given by

\[
\hat{G} (x, x_A, \omega) = \left( \begin{array}{c} \hat{G}^E_e \\
\hat{G}^H_e \\
\hat{G}^E_m \\
\hat{G}^H_m \end{array} \right) (x, x_A, \omega),
\]

where

\[
\hat{G}^E_e = \frac{\hat{E}_e^e(x_A)}{\hat{E}_e^e(x_A)}(x, x_A, \omega),
\]

\[
\hat{G}^H_e = \frac{\hat{H}_e^e(x_A)}{\hat{H}_e^e(x_A)}(x, x_A, \omega),
\]

\[
\hat{G}^E_m = \frac{\hat{E}_m^m(x_A)}{\hat{E}_m^m(x_A)}(x, x_A, \omega),
\]

\[
\hat{G}^H_m = \frac{\hat{H}_m^m(x_A)}{\hat{H}_m^m(x_A)}(x, x_A, \omega).
\]
where the superscripts \{E, H\} denote the observed field type at \(x\) and the superscripts \{e, m\} denote the source type at \(x_A\). In the submatrices each Green’s tensor denotes one 3 \(\times\) 3 Green’s tensor. Each column of \(\hat{G}\) represents a field vector at \(x\) due one particular source type and component at \(x_A\). For state \(B\) we make similar choices, replacing \(s_B\) by \(t_B(x - x_B)\) and \(u_B\) by \(\hat{G}(x, x_B, \omega)\). In case both source locations are outside \(\mathbb{D} \cup \partial \mathbb{D}\) or inside \(\mathbb{D}\) the boundary integral vanishes \([1]\). When both \(x_A\) and \(x_B\) are inside \(\mathbb{D}\) we find from substituting these choices for the sources and the fields in Equation (5) the source-receiver reciprocity relation as,

\[
\hat{G}^T(x_B, x_A, \omega) K - K \hat{G}(x_A, x_B, \omega) = 0. \tag{7}
\]

When we transpose Equation (7) and use \(KK = I\) we find the alternative expression as

\[
\hat{G}(x_B, x_A, \omega) K = K \hat{G}^T(x_A, x_B). \tag{8}
\]

Making general replacements for sources and fields Equation (5) is replaced by

\[
\hat{G}(x_B, x_A, \omega) K [\chi_\mathbb{D}(x_A) - \chi_\mathbb{D}(x_B)] = \oint_{\partial \mathbb{D}} \hat{G}(x_B, x, \omega) N_x K \hat{G}^T(x_A, x, \omega) \, d^2x, \tag{9}
\]

where Equation (8) has been used also for the Green’s functions in the integrand in the right-hand side of Equation (9). Equation (9) is an exact representation for the electromagnetic Green’s function between \(x_A\) and \(x_B\) in terms of cross-convolutions of impulsive field responses observed at the observation points \(x_A\) and \(x_B\) due to tangential electric and magnetic point sources on the boundary \(\partial \mathbb{D}\) and integrating over all source locations on the closed boundary surface \(\partial \mathbb{D}\). Possible applications of Equation (9) for electromagnetic interferometry will be investigated in the next section.

4. MODIFICATIONS FOR EM INTERFEROMETRY

In the present form, Equation (9) contains the matrix \(N_x K\) in the cross-convolution expressions in the surface integral. For a direct application in terms of convolutions of observed wave fields due to uncontrolled sources the matrix \(N_x K\) should be diagonalized, in which process a source decomposition is necessary into sources for inward and outward traveling waves and fields. We first diagonalize the representations by rewriting them in terms of observations of the electric field due to electric current sources on the boundary only. In a second step we make simplifying assumptions for the inward and outward traveling waves. These are necessary for practical transient sources.

Now we reduce the field vector to the electric field and reduce the full Green’s matrix to the electric field Green’s tensor for an electric source. Then we find \([14]\)

\[
\hat{G}^{Ee}(x_B, x_A, \omega) [\chi_\mathbb{D}(x_A) - \chi_\mathbb{D}(x_B)] = \frac{1}{j \omega \mu} \oint_{\partial \mathbb{D}} \hat{G}^{Ee}(x_B, x, \omega) \left\{n \cdot \nabla \hat{G}^{Ee}(x_A, x, \omega)\right\}^T \, d^2x - \frac{1}{j \omega \mu} \oint_{\partial \mathbb{D}} \left\{n \cdot \nabla \hat{G}^{Ee}(x_B, x, \omega)\right\} \left\{\hat{G}^{Ee}(x_A, x, \omega)\right\}^T \, d^2x, \tag{10}
\]

which is still an exact representation under the assumption that the medium in the neighborhood of the boundary is homogeneous. If we assume the points \(x_A\) and \(x_B\) lie in the far field of the boundary, then we can approximate the normal derivative and replace it by a multiplicative factor of \(\pm \sqrt{j \omega \sigma - \omega^2 \varepsilon \mu}\), where the plus sign applies to outward traveling waves and the minus sign to inward traveling waves. This multiplicative factor also assumes the contribution from fields diffusing away from the boundary in the normal direction have the major contribution in the final result. Substituting this in Equation (10) yields

\[
\hat{G}^{Ee}(x_B, x_A, \omega) [\chi_\mathbb{D}(x_A) - \chi_\mathbb{D}(x_B)] + "\text{ghost}" = -2Y \oint_{\partial \mathbb{D}} \hat{G}^{Ee}(x_B, x, \omega) \left\{\hat{G}^{Ee}(x_A, x, \omega)\right\}^T \, d^2x, \tag{11}
\]

where \(Y = \sqrt{(\sigma + j \omega \varepsilon)/(j \omega \mu)} \approx \sqrt{\sigma/(j \omega \mu)}\) denotes the complex admittance and the “ghost” term represents spurious events that are suppressed when the boundary is irregular.

We define the matrix of measured electric fields generated by transient electric current sources as

\[
\varepsilon^{\text{obs}}_{A,B}(x) = \hat{G}^{Ee}(x_{A,B}, x, \omega) \hat{S}(x, \omega), \tag{12}
\]
where $\hat{S}(x, \omega) = \text{diag}(s_1(x, \omega), s_2(x, \omega), s_3(x, \omega))$ denotes the source frequency spectrum matrix at position $x$, which can be different for each direction and for each source position. The power spectrum matrix of the sources is defined as

$$\hat{S}^p(x, \omega) = \text{diag}(|s_1(x, \omega)|^2, |s_2(x, \omega)|^2, |s_3(x, \omega)|^2).$$  \hspace{1cm} (13)

Using these definitions in Equation (11) we find

$$\hat{G}_{ee}^{\text{fe}}(x_B, x_A, \omega) \left[ \chi_D(x_A) - \chi_D(x_B) \right] \approx -2Y \int_{x \in \partial D} \hat{\varepsilon}^{\text{obs}}_B(x) \left( \hat{S}^p \right)^{-1} \left\{ \hat{\varepsilon}^{\text{obs}}_A(x) \right\}^T d^2x, \hspace{1cm} (14)$$

where the approximate sign has replaced the equality sign because we have omitted the explicit mention of the “ghost” term. The fact that the inverse of the power spectrum matrix is required indicates that it should be known to use this method for transient sources.

**ACKNOWLEDGMENT**

This work is part of the research program of the Netherlands research center for Integrated Solid Earth Science (ISES).

**REFERENCES**


