Scattering by a Finite Strip Under Complex Beam Incidence—Asymptotic Evaluation in the Complex Space Domain

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Abstract—This communication deals with the resolution of electromagnetic scattering problems by applying complex spaces. The particular problem presented is the scattering by a finite strip under complex beam incidence. It is resolved by the asymptotic evaluation of the radiation integral in the complex space domain. In addition, the outlook for the possibilities of this technique is presented: a summary of proposed problems so far, the potential and limitations.

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1. INTRODUCTION

Some classic techniques for EM problems resolution are based on complex variable. For instance, conformal mapping between complex planes to solve problems involving the Laplace equation, or complex contour path integration to obtain Green’s function for the wave equation. The authors have been concentrated on understanding the meaning of the technique of \textit{analytical continuation} of wave solutions, the complex quantities associated to it and its application to EM problems, \cite{1}. The starting point is the analytical continuation of the Greens function for the 2D complex point source, which was proposed to obtain a Gaussian beam mathematical formulation, \cite{2,3}. This technique has been use by decades and it is still being used, \cite{4}.

Some steps have been already made satisfactorily by the authors in this matter: The rigorous characterization of (i) the \textit{complex distances} appearing when the source is located at a complex position, \cite{5} (ii) the \textit{complex angles} which are defined when any observation point is related to that complex source, \cite{6}. This constitutes a duo distances-angles which allows the real observation space characterization in terms of complex spaces.

This mathematical formulation, apparently artificial, makes sense when it is applied to the 2D radiation problem of the complex point source, that is, the analysis of the \textit{complex beams}, \cite{7}. One of the most relevant aspect of this technique, which makes it so powerful, is its capability to manage exact solutions and to parameterize the different approximations and their validity ranges. Specifically, complex beams include plane waves, cylindrical waves and Gaussian beams as particular cases contained in its formulation.

In addition, another success achieved is the spectral characterization of the complex beams, \cite{8}. This opens a whole world of possibilities, inasmuch all the resolution techniques in the spectral domain may be applied to complex beams. Even when finding the complex space domain possibilities has meant a great newness, concerning to the way of thinking and working, we do not expect, in the spectral domain, a so revolutionary results, as long as the spectral domain is complex itself, and a lot of problems resolved some time ago, \cite{9}, did already operate with the complex spectral domain means.

Coming to the scattering problems, we have already posed and solved, (wholly or partially), scattering problems evaluated by asymptotic techniques in the complex space domain. The starting point was similar problems formulated in the real space domain, \cite{10,11}. Specifically, starting from the scattering by a strip or an array of strips under a different homogeneous field illuminations. We have proposed the scattering by a strip or an array of strips under some non-homogeneous fields illuminations, \cite{12}. The first incident fields we considered were non-homogeneous plane waves and Gaussian beams. Our contribution to these problems is no relevant, as long as they may be resolved by using the classical techniques, (\cite{9,11}). The reason considering these scenarios is to calibrate the method and the results. The main subject of this communication is the scattering by a finite strip under complex beam incidence which will be analyzed in next sections.

In the light of the scenarios considered, \cite{12}, we can assert that the complex space formulations are very appropriate to analyze problems concerning to non-homogeneous fields, because they are
included in a natural way inside the formulation. The disadvantage when combining with the asymptotic evaluation of integrals is that the results are approximated at the end. The accuracy is not lost but the attractive characteristic of exact solution is lost. We wonder if we will be able to apply some exact procedure (a Sommerfeld-style) to non trivial scattering problems involving complex beams.

Figure 1: Scattering by a finite perfect conductor strip under complex beam incidence. Incident field and induced current profiles.

Figure 2: Analytical continuation of the radiation integral into the complex space domain. The integration C path is deformed into the SDP.

Disclosing one of the main conclusions of this paper, we can say that the asymptotic techniques together with the complex space analysis opens the possibility to extend the results, from 2D scattering problems involving cylindrical waves to new problems involving complex beams.

2. SCATTERING BY A FINITE STRIP UNDER COMPLEX BEAM INCIDENCE

The scenario is described in Fig. 1. The scatterer is a perfect electric conductor strip located on x axis and infinite on y direction. The incident field is radiated by a 2D point source located at \((x_s, z_s)\). The analytical continuation from the real coordinates of the source into the complex space \((x_s = x_s + ib \sin \theta_{i0}, z_s = z_s + ib \cos \theta_{i0})\) gives the representation for the incident complex beam, \((\theta_{i0} \text{ is related to the beam axis direction and } b \text{ is related to the waist width})\),

\[
\vec{E}_i = E_{iy} \hat{y} = \frac{-\omega \mu_0 I e^{-i\pi/4}}{2\sqrt{2\pi}} \frac{e^{ik_0 R_i}}{\sqrt{k_0 R_0}} \hat{y}; \quad R_i = \sqrt{(x - x_s)^2 + (z - z_s)^2}. \tag{1}
\]
The usual physical optics approximation is applied to obtain the induced currents on the perfect electric conductor strip, \( J_{y, PO}(x', z') = 2z \times \vec{H} = J_{y, PO}\hat{y} \),

\[
J_{y, PO}(x', z') = \frac{-2}{i\omega \mu_0} \frac{i k_0 (z - z_s)}{\sqrt{k_0 R_i} (x, z = 0)} e^{i k_0 R_i (x, z = 0)}. \tag{2}
\]

An example of the amplitude and phase profiles of the induced currents on the real strip points is represented in Fig. 1.

The extension from the real coordinates of the strip into the space of complex coordinates is performed,

\[
J_{y, PO}(x', z') = \frac{-2}{i\omega \mu_0} \frac{i k_0 (z' - z_s)}{\sqrt{k_0 R_i} (x, z = 0)} e^{i k_0 R_i (x, z = 0)}. \tag{3}
\]

The Green’s function of complex arguments continues also being valid solution to the wave equation,

\[
\mathbf{G}(\mathbf{r}, \mathbf{r'}) = \frac{i}{4} H^{(1)}_0(k_0 R_o) \sim \frac{e^{i\pi/4}}{2\sqrt{2\pi k_0}} \frac{e^{ik_0 R_o}}{\sqrt{R_o}}; \quad R_o = \sqrt{(x - x')^2 + (z - z')^2}. \tag{4}
\]

The radiation integral, with the extended Green’s function and the extended currents on the strip complex locations, is given by,

\[
E_o^s(x, z) = i\omega \mu_0 \int_{x_s}^{x_b} J_{y}(x') G(x, z; x', z') dx' = \frac{-e^{i\pi/4}}{2\sqrt{2\pi k_0}} \int_{x_s}^{x_b} \frac{i k_0 \cos \Theta_i e^{ik_0 R_i} e^{ik_0 R_o}}{\sqrt{R_i} \sqrt{R_o}} dx', \tag{5}
\]

where the trigonometric function of a complex angle has been introduced, \( \cos \Theta_i = (z' - z_s)/R_i \).

**3. ASYMPTOTIC ANALYSIS IN THE COMPLEX SPACE DOMAIN**

The radiation integral,

\[
E_o^s(x, z) = \int_{x_s}^{x_b} f(x') e^{k_0 q(x')} dx', \tag{6}
\]

may be asymptotically evaluated, [13], in the complex space domain, Fig. 2.

\[
\int_{x_s}^{x_b} = \int_{SDP} + \int_{C_a} - \int_{C_b} = E_{SP} + E_a + E_b. \tag{7}
\]

The integral value is made up of three contributions: one coming from the complex saddle point and two coming from the endpoints of the strip.

**3.1. Complex Saddle Point**

Applying the so called Saddle Point Equation (SPE), \( q'(x_s') = 0 \), the saddle point (SP), \( (x_s', z_s') \) is obtained. It is worth mentioning that both \( dq(x', z')/dx' = 0 \) and \( dq(x', z')/dz' = 0 \) lead to the same condition,

\[
x_s' = \frac{x R_1 + R_0 x_s}{R_1 + R_0}. \tag{8}
\]

The SP is a complex point belonging to the complex straight line which goes from the complex image source \( (x_s, -z_s) \) to the real observation point \( (x, z) \).

\[
x_s' = \frac{x z_s + x_s z}{z + z_s}; \quad b' = b \frac{\sin \Theta'' (x - x_s') + \cos \Theta'' z}{(x - x_s) \sin \Theta'' + (z + z_s) \cos \Theta''}. \tag{9}
\]

In addition, \( z_s' = 0 \) and \( \Im\{x_s'\} \) is related to \( \Im\{x_s\} \) by \( b' \) parameter.

**3.2. Complex Snell Law**

Applying the SPE, the Snell law is given by \( \sin \Theta_i = -\sin \Theta_o \). This law is interpreted by means of the mathematical base previously established, [6], in particular, the results concerning to complex angles. Basically, the real part of the complex angle gives information about the propagation direction and the imaginary part is related to the attenuation at right angles to the propagation direction. Once the equation is separated into its real and imaginary parts, the real part gives the usual direction rule for the reflected rays, that is, the usual real Snell law; and the imaginary part relates the attenuation for each saddle point, and accordingly, for each associated observation point.
3.3. Complex Rays

The complex rays must be ‘projected’ into the real propagation space leading to some results which may not be found with the conventional analysis with real variable. A complex ray, which follows a straight line in the complex space, becomes a real curved line in the real space. This real space interpretation is found out by making hyperbola matching as in the Evanescent Wave Theory, [14]. The comparison between complex rays coming from both the SP and the image source, and reaching a real observation point, are shown in Fig. 3, together with its local matching with hyperbolas.

Figure 3: A complex ray (left) is a straight line in the complex space. It goes from the image source (IS), passes through the saddle point (SP) and crosses the real space in the real observation point \((\xi, \eta)\) (beam adapted coordinates have been used). The complex ray is projected as hyperbolas in the real propagation space (right).

3.4. Scattered Field

By applying the asymptotic evaluation of integrals results to the radiation integral, [13], the scattered field is obtained. The SP contribution gives the reflected field and may be seen as the field radiated by the image point source \((x_s, -z_s)\),

\[
E_{SP} = -e^{ik_0(R_i+R_o)} \sqrt{k_0(R_i+R_o)}
\]

The complex endpoint contributions give the diffracted field by the endpoints of the strip,

\[
E_{a,b} = \sqrt{2\pi} e^{-\frac{i\pi}{4}} \cos \theta_i(a, b) \frac{e^{ik_0R_i(a,b)}}{\sqrt{R_i(a,b)}} e^{ik_0R_o(a,b)} \frac{e^{ik_0R_i(a,b)}}{\sqrt{R_o(a,b)}}
\]

4. CONCLUDING REMARKS

The scattering by a finite strip under complex beam incidence has been presented. It has been resolved by a new method based on the complex space domain. Both, the problem and the solution presented here, are a generalization of the same scattering problem when the strip is illuminated by some homogeneous waves, as long as the complex beams contain the parametrization of plane waves, cylindrical waves and Gaussian beams, which appear as particular cases included in the formulation.

The complex space analysis assumes to perform the extension of the induced currents on the scatter, from its real position into complex locations. When applying asymptotic techniques, the saddle points are straight complex. Once the mathematical analysis is performed, the most interesting point is the interpretation of the results in the real space. The interpretation of the Snell law requires the results of the complex angles analysis. Basically, the real part of the gives information about the propagation direction and the imaginary part about the attenuation, for each saddle point and its associated observation point. One of the most interesting points is the interpretation of the complex rays, which correspond to curve tracks in the real space.
The formulation provides an analytical expression for the scattered field solution. The interpretation is made in terms of a reflected field and two scattered terms associated to the strip ends. The form of the scattered field fits with the expressions corresponding to a cylindrical incident field, by replacing the real quantities with the complex ones. In spite of this, the content must be carefully evaluated, so long as the diffraction factors and all their terms include complex quantities.

The results lead to think that the solution to some scattering problems in which cylindrical waves are involved, may be generalized by considering an incident complex beam. The difficulty and, at the same time, the interest consist in the interpretation of the quantities which appear in the real propagation space.

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