Applications of Pseudo-polar FFT in Synthetic Aperture Radiometer Imaging

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Abstract—In this study we analyzed the potential use of Pseudo-Polar FFT algorithm for image reconstruction of synthetic aperture radiometer, and developed an effective method to improve the image reconstruction accuracy and computational efficiency simultaneously. The advantage of the new algorithm is that it takes pseudo-polar grid instead of Cartesian grid to perform the inverse Fourier transform, and it involves only 1-D interpolation, which leading to a more fast and accurate performance. As a critical stage, the interpolation algorithms that perform the changing from polar grid to pseudo-polar grid are present. At last, the superiority of the new algorithm is validated by numerical simulation. It is believed that the new approach may have wide-spread application in science and practice for synthetic aperture radiometer system.

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1. INTRODUCTION

The Synthetic aperture radiometer is an interferometric technology that introduced from radio astronomy in the late 1980s [1]. With the main benefits of simpler mechanical structure and high spatial resolution it is getting increasingly attractive. Recently, the NASA’s ESTAR [2] and ESA’s MIRAS [3] have been demonstrated successfully, and a C-band and X-band synthetic aperture radiometer also have been developed in china [4, 5]. Further more some institutes proposed and investigated the use of aperture synthesis techniques in millimeter wave for reconnaissance and surveillance applications [6]. All the activities set up an indication that the synthetic aperture radiometer will be dominant in the passive microwave remote sensing in the future.

The antenna array plays an important role in imaging. The antenna configurations, such as ‘T’, ‘U’ or ‘Y’ shapes and so on [7], are all have redundant baselines inevitably and lead to an enormous hardware expense. Then in order to further reduce the system complexity, a new rotate configuration that can achieve sufficient samples by rotating a simple non-redundant antenna array is proposed [8, 9], which is deemed to be the most practicable way. As the sampling data is distributed in circles, the conventional Cartesian FFT method could not be used directly. A general way is to interpolate the polar grid data to Cartesian grid. However, because of the large difference between them and the low accuracy of general 2-D interpolation methods, it would introduce large errors. It is noted that the idea of CLEAN method that broadly used in radio astronomy [10] and the Moor-Penrose inversion that used by ESTAR system [2] are not suitable for this case, because the former is based on the iterative algorithm suited for point sources image but not plane sources, and the later need complicated pre-calibrate work of hardware and has high computational complexity in 2-D situation.

In this paper we introduce the pseudo-polar FFT algorithm [11, 12] to the image reconstruction. We choose pseudo-polar grid as the halfway point instead of Cartesian grid. We interpolate the polar data to pseudo-polar grid firstly and perform the Pseudo-Polar Fourier transform at last to produce the image. As the pseudo-polar grid is more close to polar grid and only 1D interpolation is needed, and the transform can also done by 1-D FFT with complexity of $O\{N^2\log N\}$, the proposed new algorithm is more accurate and high-speed.

2. PSEUDO-POLAR FFT ALGORITHM

The Pseudo-Polar Fourier transform is based on a definition of a polar like grid constituted by concentric squares, shown as Fig. 1. The pseudo-polar grid points can be separated into two groups, basically vertical ($BV$) subsets, shown as hollow points in the figure, and basically horizontal ($BH$)
subsets, shown as solid points. When the points are defined in frequency domain of $\xi \in [-\pi, \pi]^2$, their coordinates are given by

\begin{align}
BH &= \{ \xi_y = \frac{\pi l}{N}, \xi_x = \xi_y \cdot \frac{2m}{N} | -N/2 \leq m < N/2, -N \leq l < N \} \\
BV &= \{ \xi_x = \frac{\pi l}{N}, \xi_y = \xi_x \cdot \frac{2m}{N} | -N/2 \leq m < N/2, -N \leq l < N \}
\end{align}

The Fourier transform of the $BH$ and the $BV$ points are completely parallel. Our description will refer to the $BH$ points only. With the given Cartesian data $f[k_1,k_2]$, and plugging Eq. (1a) as the frequency coordinates into the Fourier transform equation, we obtain

\begin{align}
F(\xi_p,q) &= F[m,l] = \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} f[k_1,k_2] \cdot \exp \left( \frac{2\pi k_1 lm}{N^2} + \frac{\pi k_2 l}{N} \right) \\
&= \sum_{k_1=0}^{N-1} \exp \left( -i \frac{2\pi k_1 lm}{N^2} \right) \sum_{k_2=0}^{N-1} f[k_1,k_2] \cdot \exp \left( -i \frac{\pi k_2 l}{N} \right)
\end{align}

Assuming $f[k_1,k_2]$ is zero padded to $N \times 2N$, the inner summation part of Eq. (2) will be calculated as 1-D FFT

\begin{align}
F_1[k_1,l] &\equiv \sum_{k_2=0}^{N-1} f[k_1,k_2] \cdot \exp \left( -i \frac{\pi k_2 l}{N} \right) = \sum_{k_2=0}^{2N-1} f_2[k_1,k_2] \cdot \exp \left( -i \frac{\pi k_2 l}{2N} \right)
\end{align}

Then the second summation part of Eq. (2) can be proceed as

\begin{align}
F[m,l] &= \sum_{k_1=0}^{N-1} F_1[k_1,l] \cdot \exp \left( - \frac{2\pi k_1 m}{N} \cdot \frac{l}{N} \right)
\end{align}

which is same as regular 1D-FFT with a factor $\alpha = 1/N$ in its exponent part. This is known as the Fractional Fourier Transform, and can be computed efficiently with $30N \log N$ operations based on 1D-FFT. Adding to the previous stage complexity and cover both the $BH$ and $BV$ data, the whole forward transform of Pseudo-Polar Fourier transform is need $140N^2 \log N$ operations.

Figure 1: Schematic diagram of the pseudo polar grid ($N = 6$).

Figure 2: Schematic diagram of the interpolation stages.

The Pseudo-Polar FFT can be inversed by the method of Least-Squares (LS) with Conjugate Gradient (CG) algorithm. The optimization problem can be approached iteratively by

\begin{align}
f_{k+1} &= f_k - D \cdot T_{PP}^H(T_{PP}(f_k) - F)
\end{align}
where \( T_{PP} \) is the forward transform, \( D \) is the matrix that will speed up the convergence to the true solution, \( T_{PP}^H \) is the adjoint transform, theoretically same as \( T_{PP} \), referring to BV it can be compute by

\[
\hat{f}[k_1, k_2] = \sum_{l=-N}^{N-1} \exp\left(\frac{i \pi k_2 l}{N}\right) \sum_{m=-N/2}^{N/2-l} F[m, l] \cdot \exp\left(\frac{i \cdot 2\pi k_1 l m}{N^2}\right)
\]

which could be done in \( O\{N^2 \log N\} \) operations. It can achieve high accuracy solution with only 2–6 iterations.

3. IMAGE RECONSTRUCTION ALGORITHM BASED ON PSEUDO-POLAR FFT

The fundamental theory of synthetic aperture imaging is based on the Van Cittert-Zernike theory. In summary, the receivers measures the cross correlations between all pairs of antennas to get the so-called visibility function

\[
V_{kl}(u_{kl}, v_{kl}) = \frac{1}{\sqrt{\Omega_1 \Omega_2}} \int \int \frac{T_B(\xi, \eta)}{\sqrt{1 - \xi^2 - \eta^2}} \cdot F_{nl}(\xi, \eta) \cdot r_{kl}(-\frac{u_{kl} \xi + v_{kl} \eta}{f_0}) \cdot e^{-j2\pi(u_{kl} \xi + v_{kl} \eta)} d\xi d\eta
\]

where: \( \Omega \) is antenna solid angle, \( T_B(\xi, \eta) \) is brightness temperature, \((u_{kl}, v_{kl}) = (x_k - x_l, y_k - y_l)/\lambda \) is normalized antennas spacing that measured in wavelengths, \((\xi, \eta) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi)\) is direction cosines, \( F_n \) is normalized antenna voltage pattern, \( r_{kl} \) is the so called fringe-wash function, \( r_{kl} \approx 1 \) for limiting narrow band case. So for an ideal interferometer the visibility function and the brightness temperature can be associated by Fourier transform.

The sampling data measured by the rotate scanning synthetic aperture radiometer is laid in circle polar grid, which could not be inversed through Cartesian IFFT directly. As described in the previous session we use Pseudo-Polar instead of Cartesian to be the interpolated destination grid. The 1-D interpolation involved can be separated into two steps, as shown in Fig. 2, the angular interpolation that rotate the sampling points, and the radial interpolation that square the circles.

The sampling data in a circle and be deemed as a periodic function with a period of \( 2\pi \), so the angular interpolation problem can be solved by the periodic function sampling theorem, according to which, when given the periodic function \( x(t) \) with highest frequency \( f_{max} = K/T \), it can be reconstructed exactly from \( N \geq 2K + 1 \) of its uniformly spaced samples. The proof of this theorem is given in [13]. When we assume the visibility function \( F(\rho, \theta) \) is angular band limited to \( K \), it could be reconstructed by \( 2M \geq 2K + 1 \) sampling points according to

\[
F(\rho_n, \theta) = \sum_{l=0}^{2M-1} F(\rho_n, \frac{\pi l}{M}) \sin \left[ \frac{1}{2} (2M-1) \left( \theta - \frac{\pi}{M} \right) \right] 2M \sin \left[ \frac{1}{2} \left( \theta - \frac{\pi}{M} \right) \right]
\]

It’s noted that \( F(\rho, \theta) \) is not angular band limited in most cases, therefore the interpolation will produce certain artifacts in the reconstructed image. The errors can be controlled by increasing the sampling density.

The radial interpolation implemented after angular interpolation is the last stage to get the pseudo-polar grid. As it is known that the spatial domain corresponds to the sampling frequency is limited to a round disc of unit radius, \( F(\rho, \theta) \) is band limit to 2 in \( \rho \) direction, and can be reconstructed exactly according to Shannon theory. But the sampling distance in \( \rho \) is not uniformly spaced in some case and the sampling length is limited. Then the uniform Sinc interpolation could not be used. However, the band limitation property implies the relatively smoothness of \( F(\rho, \theta) \) in radial directions, a perfect interpolation is possible with appropriate samples. We considered the nonuniform Lagrange interpolation algorithm as the radial interpolation, which is defined as

\[
F(\rho_n, \theta) = w(\rho) \sum_{i=0}^{n} F(\rho_i, \theta_n) \frac{w(\rho)}{(\rho - \rho_i)w'(\rho_i)}
\]

where \( w(\rho) = (\rho - \rho_0) \ldots (\rho - \rho_n) \). Empirical studies indicated it has a good performance for radial interpolation.
4. RESULTS

We apply this new algorithm in an assumed imager system with 8-elements antenna array. There are 28 nearly even baselines altogether [14], shown as Fig. 3. In order to ensure no errors in the forward process of the simulation, we choose the Shepp-Logan phantom as the initial spatial image. Its frequency values can be calculated exactly. The Mean Square Errors (MSE) of the angular interpolations in each baseline is shown in Fig. 4, and the relative error of radial interpolation for the ray of 90° is shown in Fig. 5. We can see that the worst case error is lower than $10^{-4}$ for angular interpolation and $10^{-3}$ for radial interpolation. The best case error is about $10^{-17}$ for angular interpolation and $10^{-14}$ for radial interpolation. They are all in the acceptable field.

![Figure 3: Baselines of 8 antenna array.](image)

![Figure 4: Errors of angular interpolation.](image)

![Figure 5: Errors of radial interpolation.](image)

<table>
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<th>Methods</th>
<th>Error (MSE)</th>
<th>Similarity</th>
<th>Running time</th>
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<td>0.1890</td>
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<td>B-spline</td>
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The next stage is taking Pseudo-Polar IFFT to obtain the final image. The result is shown as Fig. 6(b). Other Cartesian FFT based methods are also applied, such as the Delannay triangulation based linear interpolation and the Biharmonic Spline interpolation. The results are shown as
Fig. 6(c) and Fig. 6(d). Table 1 exhibits the parameters of MSE, correlation coefficient (as a token of similarity) and running time of each method.

![Figure 6: The initial image (a) and the reconstruction results by the new algorithm (b), Linear (c), and B-spline (d).](image)

The simulations show that the linear method has the worst image quality. The B-spline method needs a long running time. The new method is not only has the best image quality but also the fast running speed.

5. CONCLUSIONS

In this article we developed a fast and accurate method for the rotate scanning synthetic aperture radiometer imaging. Since the frequency domain is sampled in polar coordinates, we interpolated the sampling data to pseudo polar grid by 1-D interpolation algorithm and applied Pseudo-Polar IFFT algorithm to produce the image. The performance of the new algorithm was demonstrated with a simulated 2-D scene for a given imager system. Both from the theoretical and empirical view points, the new approach is not only efficient with complexity of $O(N^2 \log N)$, but also more accurate than the known state-of-the-art methods based on Cartesian FFT. Furthermore, the interpolation stages were analyzed which plays the key role in the image reconstruction. Future work might consider various methods for improvements.

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REFERENCES


