Stochastic Perturbation of Parabolic Law Optical Solitons

Anjan Biswas
Department of Applied Mathematics and Theoretical Physics, Delaware State University
Dever, DE 19901-2277, USA

Abstract—The stochastic perturbation of optical solitons due to parabolic law nonlinearity is studied in this paper with the aid of soliton perturbation theory. The corresponding Langevin equation is derived and it is proved that the soliton propagates with a fixed mean velocity.

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1. INTRODUCTION

The dynamics of solitons propagating through optical fibers is governed by Nonlinear Schrödinger Equation (NLSE) [2]. It is well known that the pulse dynamics are not accurately described the pure NLSE. It is therefore necessary, from practical consideration, to take into account the perturbation terms.

Besides the deterministic type perturbations one also needs to take into account, from realistic situations, stochastic type perturbation. These effects can be accounted from three basic sources [2]:

1. Stochasticity associated with the chaotic nature of the initial pulse due to partial coherence of the laser generated radiation.
2. Stochasticity due to random nonuniformities in the optical fibers like the fluctuations in the values of dielectric constant the random variations of the fiber diameter and more.
3. The chaotic field caused by a dynamic stochasticity might arise from a periodic modulation of the system parameters or when a periodic array of pulses propagate in a fiber optic resonator.

Thus, stochasticity is inevitable in optical soliton communications. In this paper the NLSE with parabolic law nonlinearity is going to be studied in presence of deterministic as well as stochastic perturbation terms. The dimensionless form of NLSE with parabolic law nonlinearity is given by

\[ iq_t + \frac{1}{2} q_{xx} + (|q|^2 + \nu |q|^4) q = 0 \]

whose solution is given by

\[ q(x, t) = \frac{A}{[1 + a \cosh \{ B(x - \bar{x}(t)) \}]^2} e^{i(-\kappa x + \omega t + \sigma_0)} \]

where

\[ \frac{d\bar{x}}{dt} = v = -\kappa \]
\[ \omega = \frac{A^2}{4} - \frac{\kappa^2}{2} \]
\[ B = \sqrt{2A} \]
\[ a = \sqrt{1 + \frac{4}{3} \nu A^2} \]

Here, \( A \) is the amplitude of the soliton, \( B \) is the width, \( \kappa \) is the frequency, \( \omega \) is the wave number, \( \bar{x} \) is the center position of the soliton and \( \sigma_0 \) is the center of phase of the soliton.

Considering the effects of perturbation on the propagation of solitons through optical fibers, (1) is modified to [1]

\[ iq_t + \frac{1}{2} q_{xx} + (|q|^2 + \nu |q|^4) q = i\epsilon R \]
where
\[ R = \delta |q|^2 q + \beta q_{xx} - \gamma q_{xxx} + \lambda (|q|^2 q)_x + \nu (|q|^2)_x q + \sigma(x,t) \] (8)

For the perturbation terms, \( \delta < 0 \) is the nonlinear damping coefficient, \( \beta \) is the bandpass filtering term. Also, \( \lambda \) is the self-steepening coefficient for short pulses, \( \nu \) is the higher order dispersion coefficient and \( \gamma \) is the coefficient of the third order dispersion.

The amplifiers, although needed to restore the soliton energy, introduces noise originating from amplified spontaneous emission (ASE). To study the impact of noise on soliton evolution, the evolution of the mean free velocity of the soliton due to ASE will be studied in this paper. In case of lumped amplification, soliton are perturbed by ASE in a discrete fashion at the location of the amplifiers. It can be assumed that noise is distributed all along the fiber length since the amplifier spacing satisfies \( z_a \ll 1 \). In (8), \( \sigma(x,t) \) represents the Markovian stochastic process with Gaussian statistics and is assumed that \( \sigma(x,t) \) is a function of \( t \) only so that \( \sigma(x,t) = \sigma(t) \). Now, the complex stochastic term \( \sigma(t) \) can be decomposed into real and imaginary parts as [3]
\[ \sigma(t) = \sigma_1(t) + i\sigma_2(t) \] (9)
is further assumed to be independently delta correlated in both \( \sigma_1(t) \) and \( \sigma_2(t) \) with
\[ \langle \sigma_1(t) \rangle = \langle \sigma_2(t) \rangle = \langle \sigma_1(t)\sigma_2(t') \rangle = 0 \] (10)
\[ \langle \sigma_1(t)\sigma_1(t') \rangle = 2D_1\delta(t-t') \] (11)
\[ \langle \sigma_2(t)\sigma_2(t') \rangle = 2D_2\delta(t-t') \] (12)
where \( D_1 \) and \( D_2 \) are related to the ASE spectral density. In this paper, it is assumed that \( D_1 = D_2 = D \). Thus,
\[ \langle \sigma(t) \rangle = 0 \] (13)
and
\[ \langle \sigma(t)\sigma(t') \rangle = 2D\delta(t-t') \] (14)
In soliton units, one gets [2],
\[ D = \frac{F_n F_G}{N_{ph} z_a} \] (15)
where \( F_n \) is the amplifier noise figure, while
\[ F_G = \frac{(G-1)^2}{G \ln G} \] (16)
is related to the amplifier gain \( G \) and finally \( N_{ph} \) is the average number of photons in the pulse propagating as a fundamental soliton.

2. MATHEMATICAL ANALYSIS

The NLSE with parabolic law nonlinearity has three integrals of motion which are energy \((E)\), linear momentum \((M)\) and the Hamiltonian \((H)\). However, in this paper, only the first two of them will serve our purpose. They are respectively by [1,2]
\[ E = \int_{-\infty}^{\infty} |q|^2 dx = \frac{A^2}{aB} F \left( 1,1,\frac{3}{2}; \frac{a-1}{2} \right) B \left( 1,\frac{1}{2} \right) \] (17)
\[ M = \frac{i}{2} \int_{-\infty}^{\infty} (q q_x^* - q^* q_x) dx = \frac{\kappa A^2}{aB} F \left( 1,1,\frac{3}{2}; \frac{a-1}{2} \right) B \left( 1,\frac{1}{2} \right) \] (18)
where $F(\alpha, \lambda; \gamma; z)$ is the Gauss’ Hypergeometric function and $B(l, m)$ is the beta function. Using these integrals of motion, one can write

$$\frac{dA}{dt} = \frac{\epsilon \beta A^{m+1}}{2m^{m+1}} F \left( m + 1, m + 1, m + \frac{3}{2}; \frac{a - 1}{2a} \right) B \left( m + 1, \frac{1}{2} \right)$$

$$+ \frac{\epsilon \sqrt{2} \beta A}{a} \left[ \kappa^2 F \left( 1, 1, \frac{3}{2}; \frac{a - 1}{2a} \right) B \left( 1, \frac{1}{2} \right) - 2 \kappa A^2 F \left( 3, 1, \frac{5}{2}; \frac{a - 1}{2a} \right) B \left( 1, \frac{3}{2} \right) \right]$$

$$+ \epsilon a^2 A \sqrt{2} \left[ \sigma_1 \int_{-\infty}^{\infty} \frac{\cos \phi}{(1 + a \cosh \tau)^{\frac{3}{2}}} dx + \sigma_2 \int_{-\infty}^{\infty} \frac{\sin \phi}{(1 + a \cosh \tau)^{\frac{1}{2}}} dx \right]$$

(19)

$$\frac{d\kappa}{dt} = -\frac{\epsilon \beta A^2}{2} F \left( 3, 2, 3; \frac{a - 1}{2a} \right) B \left( 2, 1 \right)$$

$$- \frac{\epsilon \sqrt{2}}{AE} \int_{-\infty}^{\infty} \left[ \frac{a B(\sigma_2 \cos \phi - \sigma_1 \sin \phi) \sinh \tau}{(1 + a \cosh \tau)^{\frac{3}{2}}} - \frac{2 \kappa(\sigma_1 \cos \phi + \sigma_2 \sin \phi)}{(1 + a \cosh \tau)^{\frac{1}{2}}} \right] dx$$

(20)

where $\tau = B(x - \bar{x})$ and $\phi = -\kappa x + \omega t + \sigma_0$. If the terms with $\sigma_1$ and $\sigma_2$, in (19) and (20), are suppressed, the resulting dynamical system has a stable fixed point, namely a sink. Now, linearizing the dynamical system about this fixed point gives, after simplification [3]

$$\frac{dA}{dt} = -\epsilon \left( A^{m+1} - \frac{\xi}{A} \right)$$

(21)

$$\frac{d\kappa}{dt} = -\epsilon [\kappa - \zeta (1 + A - \kappa)]$$

(22)

where $\bar{A}$ is the fixed point of the amplitude while

$$\xi = \sigma_1 \int_{-\infty}^{\infty} \frac{\cos \phi}{(1 + a \cosh \tau)^{\frac{3}{2}}} dx + \sigma_2 \int_{-\infty}^{\infty} \frac{\sin \phi}{(1 + a \cosh \tau)^{\frac{1}{2}}} dx$$

(23)

and

$$\zeta = \int_{-\infty}^{\infty} \left[ \frac{a B(\sigma_2 \cos \phi - \sigma_1 \sin \phi) \sinh \tau}{(1 + a \cosh \tau)^{\frac{3}{2}}} - \frac{2 \kappa(\sigma_1 \cos \phi + \sigma_2 \sin \phi)}{(1 + a \cosh \tau)^{\frac{1}{2}}} \right] dx$$

(24)

Equation (21) and (22) are called the Langevin equations which will now be analyzed to compute the soliton mean drift velocity of the soliton. If the soliton parameters are chosen such that $\zeta A$ is small, then (22) yields [2, 3]

$$\frac{d\kappa}{dt} = -\epsilon [\kappa - \zeta (1 - k)]$$

(25)

One can solve (25) for $\kappa$ and eventually the mean drift velocity of the soliton can be obtained. Assuming that $\sigma$ is a Gaussian stochastic variable and using the initial condition $\kappa(0) = 0$ yields

$$\langle \kappa(t) \rangle = -\frac{D}{1 - D} \left\{ 1 - e^{-\epsilon(1-D)t} \right\}$$

(26)

which leads to

$$\lim_{t \to \infty} \langle v(t) \rangle = \lim_{t \to \infty} \langle \kappa(t) \rangle = \frac{D}{1 - D}$$

(27)

Thus, for large $t$, $\langle v(t) \rangle$ approaches a constant value provided $D < 1$. Thus, the soliton mean with drift approaches a constant for a large time.

3. CONCLUSIONS

In this paper, the dynamics of optical solitons with parabolic law nonlinearity in presence of perturbation terms, both deterministic as well as of stochastic, are studied. The Langevin equation were derived and the corresponding parameter dynamics was studied. The mean drift velocity of the soliton was obtained.
REFERENCES

