**Advanced Design of Phased Array Beam-forming Networks**

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**Abstract**—Recent fundamental results [1] in the theory of linear, multi-port networks enable cost-effective, higher-reliability designs for electronically-steered phased arrays. The referenced paper documents and proves that, by including a properly designed beam-forming network, it becomes possible to feed an array and steer its beam, using a much reduced number of expensive and critical phase- and amplitude-controlled sources, while at the same time completely eliminating the adverse effects of element coupling. Those new results are based on a generalization of the classical concepts of *scalar* image impedance, and of *scalar* image-transfer function for two-port networks, to the new concepts of multidimensional image-impedance *matrix*, and of multidimensional image-transfer function *matrix* for linear multi-port networks.

1. **The Price of Performance**

   Electronically-steered phased arrays provide unsurpassed agility and high angular resolution in beam-pointing, and the capability of adaptive, multifunction performance. Such highly desirable features are however only attained at the price of high cost, extreme complexity, and limited reliability. Indeed, electronically-steered phased arrays are almost always designed as active-aperture system, that include a large number of semiconductor devices and beamsteering control-elements, embedded in the physical array structure, and closely connected with all the array radiating elements. The phased arrays used in radar systems use transmit/receive modules (T/R), essentially tiny radar, each nested behind a radiating element, in a half-wavelength square section of the total array aperture. Because of the well-known low power-efficiency of semiconductors, a large heat-flux is developed locally, thus generating a complex cooling problem. Finally, notwithstanding technology advances the semiconductor devices and beam-steering control-elements still are the most expensive components of electronically-steered phased array, and cost-effective designs would only be attained by reducing their total number. Those cost and reliability advantages are however only attainable if the structure of the beam-forming network used establishes a pattern of *synergistic connectivity*, where each controlled source simultaneously feeds all the array elements, and each array element is simultaneously fed by *all* the sources (Figures 1 and 2).

![A Clustered Phased Array](image1)

**Figure 1**: A clustered phased array providing synergistic connectivity.

![Synergistic Connectivity](image2)

**Figure 2**: The aperture field is a superposition of components.

2. **Non-symmetric Beam-forming Network**

   Such cost and complexity reductions could only be feasible by including a non-symmetric, multiport beamforming network between the reduced number of active devices, and the much larger number of array radiating elements. Such beam-forming network would necessarily be non-symmetric, because of including an *n*-port interface on the side of the active devices, and an *N*-port interface on the side of the array radiating elements,
with \( n < N \) (Figures 3 and 4). The use of a reduced number of beam-steering control-elements appears possible, by considering that current active apertures have the capability of creating a very large number of completely superfluous aperture distributions, that do not generate any practical radiation pattern. Also, the angular resolution of beam-steering could be without penalty reduced, by steering the beam in increments being only a fraction of the –3 dB beam-width.

**3. Recent Theoretical Results**

The referenced, recent fundamental results [1] in the theory of multi-port networks have been attained by introducing a generalization of the classical concept of scalar image-impedance of two-port networks, to that of image-impedance matrices for multiport networks. Similarly, the classical concept of scalar image-transfer function of two-port networks, has been generalized to that of image-transfer function matrices for multiport networks. These generalizations have made possible the design of non-symmetric beam-forming networks, that are simultaneously impedance-matched to the external environment at both interfaces, while having prescribed two-way transfer functions between two interfaces with different number of ports \((n < N)\).

**4. Image Impedance Matrices**

The first fundamental new result expresses the \( n \times n \) image-impedance matrix \( Z_{11} \) for the \( n \)-port interface-1, and the \( N \times N \) image-impedance matrix \( Z_{12} \) for the \( N \)-port interface-2, as functions of the four different-size blocks \( Z_i \) of the \((n+N) \times (n+N)\) impedance matrix of a non-symmetric, multi-port network:

\[
Z_{11} = (I_n - Z_2 \cdot Z_4^{-1} \cdot Z_3 \cdot Z_1^{-1})^{1/2} \cdot Z_1 = (I_n - P_n)^{1/2} \cdot Z_1 \\
Z_{12} = (I_N - Z_3 \cdot Z_4^{-1} \cdot Z_2 \cdot Z_1^{-1})^{1/2} \cdot Z_4 = (I_N - P_N)^{1/2} \cdot Z_4
\]

where the \( n \times n \) matrix product \( P_n \), and the \( N \times N \) matrix product \( P_N \) are given by:

\[
P_n = M_n \cdot M_N = Z_2 \cdot Z_4^{-1} \cdot Z_3 \cdot Z_1^{-1} = M_{PN} \cdot \Lambda_{Pn} \cdot M_{Pn}^{-1} \\
P_N = M_N \cdot M_n = Z_3 \cdot Z_4^{-1} \cdot Z_2 \cdot Z_1^{-1} = M_{PN} \cdot \Lambda_{PN} \cdot M_{Pn}^{-1}
\]

The partial matrix-products \( M_n \) and \( M_N \) in the expressions Eqs. (3) and (4) are defined as:

\[
M_n = Z_2 \cdot Z_4^{-1} \\
M_N = Z_3 \cdot Z_1^{-1}
\]

and the matrix products \( P_n \), and \( P_N \) are mutually related by the expression:

\[
P_N \cdot (M_N \cdot M_{Pn}) = M_N \cdot (M_n \cdot M_N) \cdot M_{Pn} = M_N \cdot P_n \cdot M_{Pn} = (M_N \cdot M_{Pn}) \cdot \Lambda_{Pn}
\]
By connecting external load-networks with internal impedance matrices \( Z_{L1} = Z_{I1} \) and \( Z_{L2} = Z_{I2} \) to the two interfaces, the two image-impedance matrices will transform to each other through the non-symmetric network:

\[
Z_{I1} = Z_1 - Z_2 \cdot (Z_4 + Z_{I2})^{-1} \cdot Z_3
\]
\[
Z_{I2} = Z_4 - Z_3 \cdot (Z_1 + Z_{I1})^{-1} \cdot Z_2
\]

5. The Block-traceless Scattering Matrix

Because of the bilateral impedance match so attained, the \((n + N) \times (n + N)\) scattering matrix \( S \) of the nonsymmetric network becomes block-traceless, with only the two rectangular blocks \( S_2 \) and \( S_3 \) being non-zero:

\[
S = \begin{bmatrix}
0 & S_2 \\
S_3 & 0
\end{bmatrix}
\]

(10)

\[
S_2 = Z_2 \cdot Z_4^{-1} \cdot \left[ I_N + (I_N - Z_3 \cdot Z_1^{-1} \cdot Z_2 \cdot Z_4^{-1})^{1/2} \right]^{-1}
\]

(11)

\[
S_3 = Z_3 \cdot Z_1^{-1} \cdot \left[ I_n + (I_n - Z_2 \cdot Z_4^{-1} \cdot Z_3 \cdot Z_1^{-1})^{1/2} \right]^{-1}
\]

(12)

6. Modal and Spectral Analysis

Two other fundamental new results express the modal matrix \( M_S \), and the spectral matrix \( \Lambda_S \) of the autonormalized (normalized to the matrices \( Z_{I1} \) and \( Z_{I2} \)), block-traceless \((n + N) \times (n + N)\) scattering matrix \( S \) as:

\[
M_S = \begin{bmatrix}
M_1 & M_2 \\
M_3 & M_4
\end{bmatrix}
\]

(13)

\[
\Lambda_S = \begin{bmatrix}
\Lambda_1 & 0 \\
0 & \Lambda_4
\end{bmatrix}
\]

(14)

The modal matrix \( M_S \) has two square diagonal blocks \( M_1 \) of size \( n \times n \), and \( M_4 \) of size \( N \times N \), and two rectangular blocks \( M_2 \) of size \( n \times N \), and \( M_3 \) of size \( N \times n \), while the blocks \( \Lambda_1 \) and \( \Lambda_2 \) are \( n \times n \), and \( N \times N \):

\[
M_1 = M_{Pn}
\]

(15)

\[
M_2 = -P_n^{-1/2} \cdot Z_2 \cdot Z_4^{-1} \cdot M_{PN}
\]

(16)

\[
M_3 = Z_3 \cdot Z_1^{-1} \cdot M_{Pn} \cdot \Lambda_{Pn}^{-1/2}
\]

(17)

\[
M_4 = M_{PN}
\]

(18)

\[
\Lambda_1 = \Lambda_{Pn}^{1/2} \cdot \left[ I_n + (I_n - \Lambda_{Pn})^{1/2} \right]^{-1} = \text{diag}(e^{-\gamma_n})
\]

(19)

\[
\Lambda_4 = -\Lambda_{Pn}^{1/2} \cdot \left[ I_N + (I_N - \Lambda_{PN})^{1/2} \right]^{-1} = \text{diag}(e^{-\gamma_N})
\]

(20)

Most remarkably, the block \( \Lambda_4 \) includes \( N - n \) identically-zero eigenvalues, that correspond to the \( N - n \) identically-zero eigenvalues of the spectral matrix \( \Lambda_{PN} \) of the matrix \( P_N \), while the remaining \( n \) eigenvalues are equal to those in block \( \Lambda_1 \), save for a sign change. The 2\( n \) non-zero eigenvalues in the spectral matrix \( \Lambda_S \), and the corresponding eigenvectors, identify the two sets of \( n \) forward, and \( n \) backward, natural transmission modes of any given non-symmetric beam-forming network, while the \( N - n \) eigenvalues, that correspond to the zero-eigenvalues in block \( \Lambda_4 \), span the null-space of the \( n \times N \) block \( S_2 \), and identify the natural cut-off modes of the network. These are the \( N - n \) voltage-wave \( a_j \) vectors of the \( N \)-port interface-2, for which the received \( b_i = S_2 \cdot a_j \) vectors of the \( n \)-port interface-1 are all identically zero.

7. The Required Impedance Matrix

The final referenced fundamental result expresses the two square blocks \( Z_1 \) of size \( n \times n \), \( Z_4 \) of size \( N \times N \), and the two rectangular blocks \( Z_2 \) of size \( n \times N \), and \( Z_3 \) of size \( N \times n \), as functions of the two required
image impedance matrices $Z_{I1}$ and $Z_{I2}$, and of the two required rectangular image-transfer function matrices $S_2$ and $S_3$:

\[
Z_1 = (I_n - S_2 \cdot S_3)^{-1} \cdot (I_n + S_2 \cdot S_3) \cdot Z_{I1}
\]

(21)

\[
Z_2 = 2(I_n - S_2 \cdot S_3)^{-1} \cdot S_2 \cdot Z_{I2}
\]

(22)

\[
Z_3 = 2(I_n - S_3 \cdot S_2)^{-1} \cdot S_3 \cdot Z_{I1}
\]

(23)

\[
Z_4 = (I_n - S_3 \cdot S_2)^{-1} \cdot (I_n + S_3 \cdot S_2) \cdot Z_{I2}
\]

(24)

REFERENCES


