Low-frequency Superprism Effect and Hybridization of Transmission-line Modes in Two- and Three-dimensional Wire Media

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Abstract

The work is dedicated to the theoretic analysis of wire media, i.e. lattices of perfectly conducting wires comprised of two or three doubly periodic arrays of parallel wires which are orthogonal to one another. An analytical method based on local field approach is used together with special variant of the method of moments. The typical dispersion diagrams and iso-frequency contours are obtained and related phenomena are discussed.

Introduction

In the recent years the periodic metallic lattices have found many applications both in optical and microwave ranges (see, for example, in [1] and [2]). However, some fundamental problems have not been resolved yet, even for typical metallic electromagnetic crystals. One of them is the problem of low-frequency spatial dispersion in wire media (WM). The low-frequency spatial dispersion of a simple wire medium (a doubly periodic regular array of parallel wires) has been studied only recently in [3]. In the paper [5] this theory is generalized for double and triple wire media. The study of spatial dispersion effects in the above mentioned variants of WM has been started in work [4]. However, this study (based on the numerical approach) is far from being complete. Our theory significantly complements the results [4]. It is analytical one, and in order to validate it, a comparison to the results from [4] is carried out.

The unit cells of lattices under study are shown in Figure 1. They are comprised of two (2d or double wire medium) or three (3d or triple wire medium) doubly periodic regular arrays of parallel infinite wires which are orthogonal to one another. The wires are assumed to be perfectly conducting. The host medium is a uniform lossless dielectric with permittivity $\varepsilon_0$ and permeability $\mu_0$. The radii of wires directed along $x$, $y$ and $z$-axes are $r_x$, $r_y$ and $r_z$, and the periods of the lattice along $x$, $y$ and $z$-axes are $a$, $b$ and $c$, respectively. The lattices are spatially shifted with respect to each other by half period.

Simple Wire Media

Before consideration of 2d and 3d wire media it is useful to remind about dispersion properties of simple wire medium consisting of single grid of parallel wires. It was studied in details in our precedent papers [3,7].

Figure 1: Unit cells of double wire medium (a) and triple wire medium (b).
The dispersion equation for simple wire medium oriented along z-axis has the next form:

\[
(k^2 - q^2) \left[ \frac{1}{\pi} \log \frac{b}{2\pi r_0} + \frac{1}{bk_x^{(0)} \cos k_x^{(0)} a - \cos q_x a} + \sum_{n \neq 0} \left( \frac{1}{bk_x^{(n)} \cos k_x^{(n)} a - \cos q_x a} - \frac{1}{2\pi |n|} \right) \right] I = 0 \quad (1)
\]

where \( q = (q_x, q_y, q_z)^T \) is wave vector of the eigenmode, \( k \) is wavenumber of the host medium, and \( k_x^{(n)} = -j \sqrt{(q_y + \frac{2\pi n}{b})^2 + q_z^2 - k^2} \). The equation (1) is a real-valued dispersion equation and it has three types of solutions:

1. Ordinary waves, in case when \( I = 0 \) in (1). They have no electric field component along wires \( (E_z = 0) \) and propagate without interaction with the lattice.

2. Extraordinary waves, in case when the expression in square brackets in (1) equals to zero. They correspond to the nonzero currents \( I \neq 0 \) and have the nonzero longitudinal component of electric field \( E_z \neq 0 \). Their dispersion properties are described in details in [7].

3. Transmission-Line Modes (TLM), in case when \( (k^2 - q_z^2) = 0 \) in (1). Those waves propagate along the wires, they are TEM waves \( (E_z = 0) \), but \( I \neq 0 \). Their dispersion equation \( q_z^2 = k^2 \) has no restriction for components \( q_x, q_y \), and the phase shift of the currents in the adjacent wires can be arbitrary [3].

Under the quasi-static limit \( k a \ll 2\pi \) and \( |q| a \ll 2\pi \), the dispersion equation for extraordinary waves transforms to

\[
q^2 = q_x^2 + q_y^2 + q_z^2 = k^2 - k_0^2 \quad (2)
\]

\[
k_0^2 = \frac{2\pi/s^2}{\log \frac{s}{2\pi r_0} + F(r)} , \quad F(r) = -\frac{1}{2} \log r + \sum_{n=1}^{+\infty} \left( \frac{\coth(\pi nr) - 1}{n} \right) + \frac{\pi r}{6} \quad (3)
\]

where \( s = \sqrt{ab} \) and \( r = a/b \).

Parameter \( k_0 \) corresponds to the effective plasma frequency of the lattice \( \omega_0 = k_0/\sqrt{\varepsilon_0 \mu_0} \). For square lattices \( a = b \) one has \( F(1) = 0.5275 \). Comparing (2) with the well-known dispersion equation of uniaxial dielectrics we obtain an effective relative permittivity \( \overline{\varepsilon} \) of 1d WM in the following form:

\[
\overline{\varepsilon} = \varepsilon_0 + x_0 + y_0 y_0 \quad (4)
\]

The dependence of dielectric permittivity on \( q_z \) given by (4) does not disappear until frequency becomes zero. This fact means that wire media has low-frequency spatial dispersion. There is no low-frequency spatial dispersion for the extraordinary waves in the only case when the wave propagates across the wires \( (q_z = 0) \). At low frequencies the propagation of those waves can be described in terms of plasma-like permittivity \( \varepsilon = 1 - k_0^2/k^2 \) (see also [8]). Relative to those waves the wire medium behaves as a cold non-magnetized plasma (a continuous dielectric medium). In other propagation directions the wire medium behaves differently. In [3] we discuss the importance of the low-frequency spatial dispersion in 1d wire media. Below we theoretically show this phenomenon in 2d and 3d WM.

**Two and Three Dimensional Wire Media**

An analytical theory based on local field approximation and describing dispersion properties of 2d and 3d wire media was proposed in [5]. It will be directly used in the present work. Also, a special variant of the method of moments developed in [6] was used for simulations of dispersion diagrams and isofrequency contours, and gave perfect coincidence with results provided by the local field approach.
The dispersion equation for 2d wire medium is as follows [5]:

\[
\begin{align*}
(k^2 - q_y^2) & \left[ \frac{1}{\pi} \log \frac{c}{2\pi r_y} + \frac{1}{ck_x} \cos k_x a - \cos q_x a + \sum_{n \neq 0} \left( \frac{1}{c\beta_y^{(n)}(a)} \cos \beta_y^{(n)}(a) - \cos q_x a - \frac{1}{2\pi|n|} \right) \right] \\
(k^2 - q_z^2) & \left[ \frac{1}{\pi} \log \frac{b}{2\pi r_z} + \frac{1}{bk_x} \cos k_x a - \cos q_x a + \sum_{n \neq 0} \left( \frac{1}{b\beta_z^{(n)}(a)} \cos \beta_z^{(n)}(a) - \cos q_x a - \frac{1}{2\pi|n|} \right) \right] = \left( \frac{4q_y^2q_z^2}{k^2bc} \sin \left( \frac{k_x a}{2} \right) \sin \left( \frac{k_x a}{2} \right) \right)^2
\end{align*}
\]

where

\[
\beta_y^{(n)} = -j\sqrt{\left( \frac{q_y + 2\pi n}{b} \right)^2 + q_x^2 - k^2}, \quad \beta_z^{(n)} = -j\sqrt{\left( \frac{q_y + 2\pi n}{c} \right)^2 + q_z^2 - k^2}.
\]

The dispersion equation for 3d wire medium is presented in [5] and has much more cumbersome form than (5). However, in the special case when \( q_z = 0 \) it splits into two separate terms: the first one is the dispersion equation for the 1d WM of \( x \)-wires and the second one is the expression (5). The first case corresponds to the extraordinary waves propagating normally to the \( x \)-wires without interaction with \( y \)- and \( z \)-wires. There is no spatial dispersion for those waves. The second case corresponds to the in-plane propagation in 2d WM, which will be studied below. In the present paper we do not consider the general case of the wave propagation in 3d WM.

At low frequencies 2d and 3d WM can be described by effective permittivity dyadics of the form [5]:

\[
\begin{align*}
\hat{\varepsilon}_{\text{double}} &= \varepsilon_{xx} \hat{x} \hat{x} + \varepsilon_{yy} \hat{y} \hat{y} + \varepsilon_{zz} \hat{z} \hat{z}, & \hat{\varepsilon}_{\text{triple}} &= \varepsilon_{xx} \hat{x} \hat{x} + \varepsilon_{yy} \hat{y} \hat{y} + \varepsilon_{zz} \hat{z} \hat{z} \\
\epsilon_{xx} &= 1 - \frac{k^2_0(r_x,b,c)}{k^2 - q_x^2}, & \epsilon_{yy} &= 1 - \frac{k^2_0(r_y,a,c)}{k^2 - q_y^2}, & \epsilon_{zz} &= 1 - \frac{k^2_0(r_z,a,b)}{k^2 - q_z^2}.
\end{align*}
\]

The effects of the low-frequency spatial dispersion are described by terms \( q_x, q_y, q_z \) in the denominators of the components of \( \hat{\varepsilon} \) (see also in [3]).

The preliminary analysis of (5) reveals some special solutions. There are two solutions which correspond to TLM: the first is \( q_y = k \), \( q_x = 0 \), \( q_z \) is arbitrary; the second one is \( q_x = k \), \( q_y = 0 \), \( q_z \) is arbitrary. They are TEM waves as well as TLM in simple WM [3]. The component \( q_x \) is a free parameter for TLM and plays a role of a phase shift between the currents in the adjacent grids of wires [3]. At the first sight, it seems strange that the electric field with non-zero \( z \)-component can propagate along \( y \)-directed wires below the “plasma” frequency which is the cut-off frequency for such waves in 1d WM. However, it is possible. When the TLM propagates along the \( y \)-wires, all grids of \( z \)-wires are excited, however the superposition of their fields exactly vanish in the planes \( x = am + a/2 \), where the grids of \( y \)-wires are located.

When \( q_x = \pi/a \) or \( q_y q_z = 0 \) the right side of (5) equals to zero and the equation splits into two separate equations similar to 1 and describing the extraordinary waves in two simple WM. For \( q_y = 0 \) (or \( q_z = 0 \)) the absence of the interaction between two simple WM is trivial since the propagation holds in the plane \( (x - z) \) or \( (x - y) \) and the electric field is polarized orthogonally to \( y \)-directed wires (or to \( z \)-wires). However, the interaction between two 1d WM is also absent when \( q_x = \pi/a \). At low frequencies \( ka < 1 \) the equation \( q_x = \pi/a \) corresponds to the excitation of TLM in both \( y \)- and \( z \)-arrays with polarization directions alternating along \( x \). The existence of this kind of TLM (which does not transport energy at all) is specific for 2d WM.

**Dispersion Diagrams and Isofrequency Contours of Double Wire Medium**

The dispersion diagram of a double WM for the in-plane propagation \( (q_x = 0) \) of the extraordinary waves obtained by numerical solution of 5 is shown in Figure 2. The chosen parameters of the wire lattice are \( a = b = c \), \( r_y = r_z \). The filling ratio is \( f = 2\pi r_y^2/a^2 = 0.002 \). We use notations \( \Gamma = (0,0,0)^T \), \( Z = (0,0,\pi/c)^T \), and \( L = (0,\pi/b,\pi/c) \) for the central point, the \( z \)-bound point and the corner point of the fundamental Brillouin zone, respectively.

One can notice the significant difference between Figure 2 and the dispersion diagram of a simple wire medium. In the Figure 2 one can see within the interval \( L - \Gamma \) two extraordinary modes which do not vanish.
Figure 2: Dispersion diagram of a double wire media with filling ratio $f = 2\pi r_0^2/a^2 = 0.002$. Thin lines - modes of the host medium (singular points of equation (5)), thick lines - modes of the 2d WM.

at low frequencies $k < k_0$ and are not TLM. In simple WM the waves with nonzero longitudinal (with respect to the wires) component of the electric field cannot propagate at low frequencies since the phase shifts between the adjacent wires are small and the re-radiation of parallel wires suppresses the wave. In 2d WM it becomes possible due to the electromagnetic interaction of the two orthogonal wire arrays. This is the result of the cross-polarized interaction of wire arrays.

Figure 3: Isofrequency contours for double wire media, a) $ka/(2\pi) = 0.1$, b) $ka/(2\pi) = 0.3$. Two cases $q_x = 0$ and $q_x = \pi/(2a)$ are presented.

The horizontal lines $ka/(2\pi) = 0.1$ and $ka/(2\pi) = 0.3$ in Figure 2 correspond to isofrequency contours presented in Figure 3.a and 3.b, respectively. The isofrequency contour located around the L point is very unusual (close to the hyperbolic one). In Figure 3.a one can see that the contours of isofrequencies are rather close to four asymptotes $q_{y,z} = \pm k$. In spite of the rather low frequency as compared to $\omega_0$, the isofrequency contour located around the $\Gamma$ point ($q = 0$) basically differs from the isofrequency of an isotropic dielectric (a circle). Only in a special case of the in-plane propagation the isofrequency centered at the $\Gamma$ point has practically circular shape and the phase velocity of this mode coincides with that of the host medium. When $q_x \neq 0$ the shape of this isofrequency becomes super-quadric and modes with hyperbolic isofrequency tend to the same asymptotes $q_{y,z} = \pm k$. When $q_x = \pi/a$ the isofrequencies coincide with the asymptotes exactly. This
case corresponds to TLM discussed above (which do not transport energy). The plot in Figure 3.a indicates the possibility of the two refracted waves (both extraordinary waves) for the rather large sheer of incidence angles. Moreover, 2d WM with proper orientation of wires with respect to the interface can possess the low-frequency negative refraction.

At the frequencies close to the plasma frequency \(\omega_0\) and higher two other modes appear with isofrequencies centered at \(\Gamma\). They are shaped as two crossing ellipses. The modes with isofrequency curves close to \(q_{y,z} = \pm k\) are still present. The isofrequency contours for such a case (corresponding to \(ka/2\pi = 0.3, q_x = 0\) and \(q_x = \pi/(2a)\)) are shown in Figure 3.b. When \(q_x\) increases at fixed frequency, the hyperbolic isofrequency contours in the plane \((q_y - q_z)\) approach the asymptotes in the same way as it happens for lower frequencies. The elliptic contours located around \(\Gamma\) (see Figure 3.b) shrink to this point when \(q_x\) grows and disappear when \(q_x\) becomes greater than \(k_0\).

**Conclusion**

In the present paper we have studied dispersion properties of double and triple wire media. We have theoretically revealed the following effects of low-frequency spatial dispersion for 2d WM. Propagation of \(z\)–polarized TLM along \(y\)–wires is not suppressed by the presence of \(z\)–wires (the same is correct for the \(y\)–polarized TLM propagating along \(z\)). There are TLM which can exist in both \(y\)– and \(z\)–arrays simultaneously. These modes do not transport energy, since the directions of the currents in wires are alternating along the \(x\)–axis. There are two propagating modes at low frequencies \(\omega < \omega_0\) which are not TLM and not ordinary waves. One mode has non-zero electric field component in the plane \((y - z)\) whereas both \(y\)– and \(z\)–components of the permitivity tensor are negative. For the other one the isofrequency contour is nearly hyperbolic. Near the plasma frequency the two other waves appear with crossing isofrequency contours. The materials under consideration could find various applications due to the properties discussed in this paper. We would like to note especially such applications as creation of low frequency super-prism and design of materials with negative refraction.

**REFERENCES**