Eigenvalue Analysis of Curved Open Waveguides Using a Finite Difference Frequency Domain Method Employing Orthogonal Curvilinear Coordinates

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Abstract

Eigenvalue analysis of open curved geometries is performed by using a two dimensional (2-D) Finite Difference Frequency Domain (FDFD) eigenvalue method employing orthogonal curvilinear coordinates, in conjunction with a perfectly matched layer (PML) tensor. This method can be used to compute the dispersion characteristics of open curved structures such as open microstrip lines printed on curved substrates. Numerical results for the eigenvalues of several geometries are presented, and compared against already published results, so as to validate the accuracy of the method.

Introduction

The numerical solution of the eigenvalue problems for open waveguide structures constitutes one of the most challenging problems of computational electromagnetism. Due to the difficulty of the problem only a few articles have been published toward this aim, e.g. [1-3]. In these works the Finite Difference Frequency Domain (FDFD) method in conjunction with the Perfect Matching Layer (PML), [4-5], technique was employed to confine the solution domain to a finite area. Even though the PML technique still suffers from spurious modes, is currently the most powerful technique for the establishment of “transparent” absorbing boundary conditions. Moreover, the anisotropic material PML formulation by Sacks et al., [5], offers the advantage that it does not need any modification of Maxwellian equations and can be implemented as a diagonal permittivity and permeability tensor.

However, to the authors knowledge, none of the published techniques can handle curved open geometries. The technique presented herein is based on a 2-D Finite Difference Frequency Domain eigenvalue method formulated in orthogonal curvilinear coordinates. Preliminary results for closed curved geometries are published in our previous work, [6]. The 2-D FDFD analysis is formulated as an eigenvalue problem for the complex propagation constants and it is restricted to structures uniform along the third direction ($u_3$-axis), along which field propagation is considered. The major advantage of this method is that the waveguiding structure can be curved in all directions. Also, a challenging problem refers to correctly handling the PML tensor along with the FDFD numerical method, so as to reduce or eliminate spurious modes and to obtain accurate numerical results for a given open curved geometry.

Formulation of the FDFD Eigenvalue for Curved Waveguides

The proposed 2D-FDFD scheme aims at the formulation of an eigenvalue problem for the propagation constant of two-dimensional structures in orthogonal curvilinear coordinates ($u_1, u_2, u_3$). The wave is assumed to propagate along the $u_3$-direction, while the cross section ($u_1, u_2$) of the waveguide structure can be of arbitrary geometry (Fig.1) loaded with inhomogeneous and in general anisotropic materials described by tensor permittivity $\bar{\varepsilon}(u_1, u_2)$ and permeability $\bar{\mu}(u_1, u_2)$, where $\bar{\varepsilon}$ is the permittivity tensor $\bar{\varepsilon} = \begin{bmatrix} \varepsilon_{tt} & \varepsilon_{tl} \\ \varepsilon_{lt} & \varepsilon_{ll} \end{bmatrix}$ and $\bar{\mu}$ the permeability tensor $\bar{\mu} = \begin{bmatrix} \mu_{tt} & \mu_{lt} \\ \mu_{lt} & \mu_{ll} \end{bmatrix}$ with $\varepsilon_{tt} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix}$ and $\varepsilon_{ll} = \begin{bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}$.

Starting from the Maxwell’s curl equations for the electric and magnetic field:

$$\nabla \times \vec{H} = j\omega \bar{\varepsilon} \vec{E} + \vec{J}$$

(1)

$$\nabla \times \vec{E} = -j\omega \bar{\mu} \vec{H}$$

(2)
Likewise, starting from (2), we have

\[ \frac{\partial}{\partial h} \text{the curve of the guide axis to shapes in which} \]

that the metric factor \( h \) must be also independent of \( u \). Have to impose certain restrictions on the metric coefficient \( s \).

Moreover, in order to simplify the problem and distinguish the transverse and longitudinal parts, the field quantities \( E, H \) and the nabla operator \( \nabla \) can be discriminated into longitudinal and transverse quantities as:

\[ \tilde{E} = E_t + \hat{a}_3 E_3, \tilde{H} = H_t + \hat{a}_3 H_3 \quad \text{and} \quad \nabla = \nabla_{tc} - (j\beta/h_3)\hat{a}_3 \]

where \( h_1, h_2, h_3 \) are the scale (or metric) factors and \( \hat{a}_1, \hat{a}_2, \hat{a}_3 \) the unit vectors.

Considering wave propagation toward the positive \( u_3 \)-direction as \( \propto e^{-j\beta u_3} \), then \( \partial/\partial u_3 \rightarrow -j\beta \), provided that the metric factor \( h_3 \) is independent of \( u_3 \), according to Lewin [8].

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where

\[ \nabla_{tc} = \hat{a}_1 \left( \frac{1}{h_1} \right) \frac{\partial}{\partial u_1} + \hat{a}_2 \left( \frac{1}{h_2} \right) \frac{\partial}{\partial u_2} \]

It is important to notice, that in order for the nabla operator to be discriminated according to Eq.(4) we have to impose certain restrictions on the metric coefficients \( h_1 \) and \( h_2 \). Again, according to Lewin [8], these must be also independent of \( u_3 \), namely \( \partial h_1/\partial u_3 = 0, \partial h_2/\partial u_3 = 0 \).

Considering now that the curl operator \( \nabla \times (\cdot) \) can be written

\[ \nabla \times \tilde{E} = \nabla_{tc} \times \tilde{E} - (j\beta/h_3)\hat{a}_3 \times \tilde{E} \]

after some algebraic operations we conclude to:

\[ \nabla \times \tilde{E} = -j\omega \tilde{H} \rightarrow \begin{pmatrix} -j\beta \cdot \left( \frac{1}{h_3} \right) \cdot \hat{a}_3 \times (\cdot) \\ -\nabla_{tc} \cdot \hat{a}_3 \times (\cdot) \end{pmatrix} = -j\omega \begin{pmatrix} \tilde{E}_t \\ h_3 E_3 \end{pmatrix} \]

Likewise, starting from (2), we have

\[ \nabla \times \tilde{H} = j\omega \tilde{E} \rightarrow \begin{pmatrix} -j\beta \cdot \left( \frac{1}{h_3} \right) \cdot \hat{a}_3 \times (\cdot) \\ -\nabla_{tc} \cdot \hat{a}_3 \times (\cdot) \end{pmatrix} = j\omega \begin{pmatrix} \tilde{H}_t \\ h_3 H_3 \end{pmatrix} \]

The form of Maxwell equations (7) and (8) is appropriate for discretization with the aid of a curvilinear grid. In this manner these will be applied to the whole solution domain. The resulting system of equations represents the eigenvalue formulation provided that the boundary conditions are incorporated into this system by proper modification of the matrices involved.
An Eigenvalue-based method aims at the transformation of problems such as (7) and (8) to nondeterministic eigenproblem of the form:

$$[A][u] = \beta[u]$$  \hspace{1cm} (9)

Matrix $[A]$ consists of sub-matrices, which represent the discrete form of the basic operators, [9], such as gradient and divergence. Matrix $[u]$ is the eigenvector and $\beta$ the sought eigenvalue. Each operator is formulated in a matrix form and modifies the problem into a simple matrix multiplying problem. The form of these operators depends on the approach used for the discretization of the fields. In our case the solution domain is discretized to form an orthogonal curvilinear mesh according to the basic principles of Yee’s mesh, [10], and every operator is formed with respect to that mesh, as in Fig.2, where a curvilinear cell is shown. Its corresponding Discretized Curvilinear Gradient Operator is shown in equation 10.

$$G_{cct} = \begin{bmatrix}
-(\Delta_1 h_1 \Delta u_1)^{-1} & (\Delta_1 h_1 \Delta u_1)^{-1} & 0 & 0 \\
0 & 0 & -(\Delta_2 h_1 \Delta u_1)^{-1} & (\Delta_2 h_1 \Delta u_1)^{-1} \\
0 & -(\Delta_2 h_2 \Delta u_2)^{-1} & 0 & 0 \\
0 & 0 & 0 & (\Delta_2 h_2 \Delta u_2)^{-1}
\end{bmatrix}$$  \hspace{1cm} (10)

The eigenvalue problem defined by (7) and (8) can be transformed now to its discretized form as:

$$\begin{pmatrix}
-(j\beta) \cdot (\frac{1}{\eta_0}) \cdot A_e & -A_e \cdot (\frac{1}{\eta_0}) \cdot G_{cct} \\
-D_{cmt} \cdot A_e & 0
\end{pmatrix}
\begin{pmatrix}
\bar{E}_t \\
h_3 E_3
\end{pmatrix}
= -j\omega
\begin{pmatrix}
M_{tt} & M_{lt} \\
M_{lt} & M_{ll}
\end{pmatrix}
\begin{pmatrix}
\bar{H}_t \\
h_3 H_3
\end{pmatrix}$$  \hspace{1cm} (11)

$$\begin{pmatrix}
-(j\beta) \cdot (\frac{1}{\eta_0}) \cdot A_m & -A_m \cdot (\frac{1}{\eta_0}) \cdot G_{cmt} \\
-D_{cmt} \cdot A_m & 0
\end{pmatrix}
\begin{pmatrix}
\bar{H}_t \\
h_3 H_3
\end{pmatrix}
= j\omega
\begin{pmatrix}
E_{tt} & E_{lt} \\
E_{lt} & E_{ll}
\end{pmatrix}
\begin{pmatrix}
\bar{E}_t \\
h_3 E_3
\end{pmatrix}$$  \hspace{1cm} (12)

Subscript e and m denotes operators acting on electric and magnetic field components respectively. These are different since they are discretized on different grids (shifted by half cell), [9-10]. Subscript c indicates that all the operators have been discretized to form a curvilinear problem, while subscript t denotes a transverse operator and l denotes a longitudinal operator. For instance, $G_{cct}$ and $G_{cmt}$ are the transverse curvilinear Gradient operators corresponding to the electric field (as shown in Fig.2) and magnetic field respectively. Moreover, $D_{cct}$ and $D_{cmt}$ are the transverse curvilinear Divergence operators corresponding to the electric and magnetic field respectively. Matrices $A_e$ and $A_m$ represent the curl operators $\hat{a}_3 \times (\hat{E}_t)$ and $\hat{a}_3 \times (\hat{H}_t)$ respectively. Finally submatrices E and M are the discretized permittivity and permeability tensors as shown in Fig.1. The eigenproblem $[A][u] = \beta[u]$ can be extracted now through (11) and (12) and due to the sparcity of matrices involved, it can be solved using the Arnoldi Algorithm, [11].

**Implementation of the Anisotropic PML**

As the FDFD code has the capability to handle anisotropic lossy media a PML is easy to implement straightforward with the introduction of the corresponding well known permittivity and permeability tensor, of the form, [2]:

PML:

$$[\varepsilon] = \varepsilon_0 \begin{bmatrix}
\eta & 0 & 0 \\
0 & \eta^{-1} & 0 \\
0 & 0 & \eta
\end{bmatrix}$$ and $$[\mu] = \mu_0 \begin{bmatrix}
\eta & 0 & 0 \\
0 & \eta^{-1} & 0 \\
0 & 0 & \eta
\end{bmatrix}$$  \hspace{1cm} (13)

where $\eta = 1 - j\zeta$ and $\zeta = (\sigma_u/\omega\varepsilon_0)$. These tensors are valid for a PML acting in the y-direction as is shown in Fig.5. The electric conductivity in the PML is assigned, [1], as

$$\sigma_u = \sigma_m \left(\frac{y}{d}\right)^n$$ where $\sigma_m = \frac{(n + 1)\varepsilon_0 c}{2d} \ln R_{th}$  \hspace{1cm} (14)

where $d$ is the PML thickness, $\sigma_m$ is the maximum electrical conductivity at the outer side of the PML and $R_{th}$ the theoretical Reflection coefficient.
Numerical Results

Numerical tests are carried out to verify the proposed method. In the first example a shielded coupled microstrip line with finite thickness, shown in Fig.3(a), is analyzed. The numerical results obtained for even and odd quasi-TEM modes are compared against the corresponding results in [1]. Very good agreement is reached as shown in Fig.4(a). The next step is the study of a curved parallel coupled microstrip line shown in figure 3(b), with the same cross section as in Fig.3(a) example. For the simulation of the curved structure it is assumed that \( \mathbf{u}_1 = \hat{x}, \mathbf{u}_2 = \hat{y}, \mathbf{u}_3 = \hat{s} \) and \( h_1 = 1, h_2 = 1, h_3 = 1 + y/R \), [12], where \( R \) is the curvature radius. As it is noticed from Fig.4(b) the eigenvalues for even or for odd modes were increased at about 15%, when the curvature radius \( R \) was 0.5m, with respect to the straight lines.

Figure 3: a) Cross Section of shielded parallel coupled microstrip lines with finite strip thickness. b) Geometry of parallel coupled microstrip lines printed on a curved substrate.

The open microstrip line shown in Fig.5 is studied in our 3\textsuperscript{rd} example and the PML tensor impact on accuracy is examined. It has been found in [1], that the theoretical Reflection coefficient \( R_{th} \) and power \( n \) used in Eq.(14) must be correctly chosen in order to achieve small PML reflections.

For this problem the \( R_{th} \) is chosen to be \( 10^{-9} \) and \( n = 2.7 \), according to [1]. The PML thickness \( d \) is 10 cells of 0.15mm each. As shown in Fig.5 the PML tensor given in Eq.(14) is employed only in the y-direction, while a large dimension in the x-direction without a PML gave accurate results. In general, the implementation of PML in all directions is expected to be necessary. Table I presents a comparison of our results against those given in [1]. The small imaginary part of the eigenvalues is an error introduced by the losses involved in the PML tensors. In turn, the same open microstrip line is studied considering a curved substrate. An increase of the propagation constant of about 2.5-5% is observed again. This is a smaller increase compared to the shielded case of Fig.4(b), because the structure is open in the y-direction.

Figure 4: Dispersion curves of quasi-TEM modes for shielded parallel coupled microstrip lines for two geometries: a) Straight as in fig. 3(a) and comparison against [1] and b) Curved structure with radius \( R=0.5m \) as in fig. 3(b).

Figure 5: Cross section of the computation domain for an open microstrip line. \( \varepsilon_r=9.4, \ w=0.5 \ mm, \ h=1.5mm \).
Table 1: Results for a straight and curved open microstrip line for two curvature radius

<table>
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<th>F(GHz)</th>
<th>F(GHz)</th>
<th>Reference Data(ε_{eff})</th>
<th>Present Method</th>
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<td></td>
<td></td>
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<td>PML</td>
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<tr>
<td></td>
<td></td>
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</tr>
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</table>

Conclusion

In this paper, an eigenvalue analysis of curved geometries both open and closed is established using a 2-D FDFD method. This was formulated in orthogonal curvilinear coordinates in conjunction with a perfectly matched layer (PML) tensor. Numerical results have been presented to demonstrate the validity of the method for both closed-shielded and open-radiating geometries either straight or curved. It was also observed that when the direction of propagation is curved the eigenvalues are increased for both closed and open waveguiding structures.

REFERENCES