A Robust Preconditioner for GMRES Method Applied to Finite Network Method

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Abstract

The aim of the authors is the calculation of induction and proximity effects of electric machines in an intermediate frequency range up to 100 kHz by the Finite Network Method (FNM). FNM excludes non-magnetic and non-conductive media (e.g. air) from the numerical calculation and fulfills the asymptotic boundary conditions for the free space automatically. We present a short derivation of FNM for non-magnetic materials from Maxwell’s equations.

Furthermore, we present the applicability of the GMRES method in the whole frequency range. In the low frequency limit $f \to 0$ the impedance matrix reduces to its real part and can be used as preconditioner for GMRES. If the preconditioner is diagonally dominant preconditioning can be performed by Gauss-Seidel iterations.

As an example we calculated the skin and proximity effect in two parallel copper wires with 200 to 500 degrees of freedom. Our problem allows the iterative calculation of the preconditioner by the Gauss-Seidel method. The preconditioner results in an acceleration of GMRES up to a factor 8 in addition with an early restart of the preconditioned GMRES.

Despite the considerable portion of the imaginary part of the system matrix at higher frequencies the Gauss-Seidel preconditioner is applicable in the whole frequency range.

Introduction

The calculation of inductance phenomena is of main interest for the simulation of electric machines. Most numerical methods such as FDM and FEM use the magnetic vector potential formulation to solve the electromagnetic field problem. Unfortunately, FDM and FEM need the discretization of the whole area of interest. This includes a large cover of free space with appropriate boundary conditions in infinity to simulate the far field.

An alternative method ([1],[2]) is the formulation of the field problem by the Finite Network Method (FNM). The FNM excludes non-magnetic and non-conductive media from the simulation. As the main advantage of FNM this method leads to a linear system of equations which is small compared to the usual application of FDM and FEM.

1. Theory of Finite Network Method (FNM)

We consider the system of Maxwell’s equation in the quasistatic case and in the absence of a free charge density ($\rho = 0$). For simplicity, we apply the FNM only to non-magnetic materials. The magnetic field $\vec{H}$ can be replaced by the magnetic flux density $\vec{B} = \mu_0 \vec{H}$. We use the magnetic vector potential in Coulomb gauging:

$$\vec{B} = \text{rot}\vec{A}, \quad \text{div}\vec{A} = 0$$

(1)

The magnetic vector potential is determined by its sources, the distribution of electric currents $\vec{J}$:

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{4\pi} \int_V dv' \frac{\vec{J}(\vec{r}', t)}{|\vec{r} - \vec{r}'|}$$

(2)

Faraday-Lorentz law of Maxwell’s equations $\text{rot}\vec{E} = \text{rot}[-\frac{\partial}{\partial t}\vec{A}]$ results in

$$\vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t}\vec{A}(\vec{r}, t) - \text{grad}\varphi(\vec{r})$$

(3)
with $\varphi(\vec{r})$. The material law $\vec{E} = \frac{1}{\gamma(\vec{r})} \vec{J}$ is applied to (3):

$$- \text{grad} \varphi(\vec{r}) = \frac{1}{\gamma(\vec{r})} \vec{J} + \frac{\mu_0}{4\pi} \int_{V'} dV' \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial}{\partial t} \vec{J}(\vec{r}', t)$$  \hspace{1cm} (4)

In the Finite Network Method (FNM, see fig. 1) we establish closed loop integrals to equation (4). $U^{(q)}(\xi)$ is the sum of all voltage sources of loop $C_\xi$:

$$U^{(q)}(\xi) = \oint_{C_\xi} d\vec{r} \frac{1}{\gamma(\vec{r})} \vec{J}(\vec{r}, t) + \frac{\mu_0}{4\pi} \oint_{C_\xi} \oint_{V'} dV' \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial}{\partial t} \vec{J}(\vec{r}', t)$$  \hspace{1cm} (5)

In the discretisation scheme of FNM we solve the problem in the following manner: Conductive volumes are discretised into small hexagonal elements carrying three constant components of the electric current (see fig. 1). Every isolated conductive volume is described by independent meshes identical to the closed loops of equation (5). The system of elementary current components has to be transformed into an equivalent system of mesh currents $i^{(m)}$ using the graph of the network. An averaging of the cross section $q_\xi$ of loop $C_\xi$ is added to the closed loop integrals of (5). Equation (5) can be rewritten in the harmonic case:

$$\mu_0^2 U^{(q)}(\xi) = \sum_\eta R_{\xi,\eta} i^{(m)}(\eta) + j\omega \sum_\eta \frac{\mu_0}{4\pi} \int_{q_\xi} \int_{q_\eta} dq_\xi dq_\eta \oint_{C_\xi} \oint_{C_\eta} \frac{d\vec{l}_\xi d\vec{l}_\eta}{|\vec{r}_\xi - \vec{r}_\eta|}$$  \hspace{1cm} (6)

The resistive parts of (6) are calculated as follows:

$$R_{\xi,\eta} = \begin{cases} \oint_{C_\xi} \frac{d\vec{l}_\eta}{\gamma(\vec{r}_\eta)} & \xi = \eta, \text{ closed loop } C_\eta \\ \pm \oint_{C_\xi \cap C_\eta} \frac{d\vec{l}_\eta}{\gamma(\vec{r}_\eta)} & \xi \neq \eta, \text{ common branch of meshes } C_\xi \text{ and } C_\eta \\ 0 & \text{no common branch} \end{cases}$$  \hspace{1cm} (7)

The sign of the resistors $R_{\xi,\eta}$ in (7) in the case $\xi \neq \eta$ depends on the orientation of the mesh currents $i^{(m)}_\xi$ and $i^{(m)}_\eta$ which is identical to the orientation of loops $C_\xi$ and $C_\eta$. In the second term of (6) you can define the inductances $L_{\xi,\eta}$ between mesh currents $i^{(m)}_\xi$ and $i^{(m)}_\eta$:

$$L_{\xi,\eta} = \frac{\mu_0}{4\pi} q_\xi q_\eta \int_{q_\xi} \int_{q_\eta} dq_\xi dq_\eta \oint_{C_\xi} \oint_{C_\eta} \frac{d\vec{l}_\xi d\vec{l}_\eta}{|\vec{r}_\xi - \vec{r}_\eta|}$$  \hspace{1cm} (8)

The inductances (8) include an averaging over the cross section of the loops. Equation (6) results in an linear equation system:

$$U^{(q)}(\xi) = \sum_\eta R_{\xi,\eta} i^{(m)}(\eta) + j\omega \sum_\eta L_{\xi,\eta} i^{(m)}(\eta)$$  \hspace{1cm} (9)
2. Algorithms for Linear Equation System Solvers

The GMRES method (Generalized Minimum Residual) [4] is an iterative equation solver. In our calculations we made use of GMRES with right preconditioning (PGMRES) [5]. The PGMRES method solves a linear equation system (10) with non-Hermitian \( n \times n \) complex system matrix \( A \) and the preconditioner \( M \). With the starting residual (11) the Krylov subspace \( \kappa_m \) is constructed at iteration step \( m \) (see (12)):

\[
AM^{-1}z = b \quad \text{with} \quad x = M^{-1}z
\]  
(10)

\[
r_0 = b - Ax_0
\]  
(11)

\[
\kappa_m = \text{span}\{r_0, AM^{-1}r_0, \ldots, (AM^{-1})^{m-1}r_0\}, m < n
\]  
(12)

During each step \( m \) the functional

\[
R(z) = \|b - AM^{-1}z\|_2
\]

is minimized in \( \kappa_m \). We used the classical Gram-Schmidt variant of Arnoldi algorithm to construct an orthonormal basis of \( \kappa_m \) and solve the extremal problem on QR factorization using Givens rotations. The unique minimum \( z_{min} \) at iteration step \( m \) provides the approximate solution of the preconditioned equation system (10):

\[
x_m = x_0 + M^{-1}z_{min}
\]  
(13)

In all cases we use the starting vector \( x_0 = 0 \). Furthermore, the above algorithm provides the residual \( r_m \) of iteration step \( m \) automatically without an explicit calculation of (15):

\[
r_m = b - Ax_m
\]  
(14)

The algorithm is stopped when the criteria

\[
\|r_m\|_2 \leq tol
\]  
(15)

is fulfilled. Restarting with the last solution \( x_m \) is applied when the stopping criteria is not fulfilled after a defined number of iterations \( m_{max} \). A modification of the PGMRES solver is the enforcement of the restarting by a significant small value of \( m_{max} \) (early restarting).

Let us consider the system matrix \( M \) of (9) in the low frequency limit \( f \rightarrow 0 \):

\[
M = [R_{\xi,\eta}]
\]  
(16)

\( M \) only contains the mesh resistors \( R_{\xi,\eta} \) of the electric network and is used as preconditioner for GMRES. The application of the current mesh method to our example of parallel wires results in an diagonally dominant and real \( M \). The calculation of matrix-vector products \( y = M^{-1}z \) is performed by the solution

of the following equation system:

\[
My = z
\]  
(17)

Since \( M \) is diagonally dominant equation (17) can be solved by the Gauss-Seidel method. The Gauss-Seidel method will converge for an arbitrary starting vector (see [6]). The iterative solution of (17) is numerically inexpensive because of the sparse structure of \( M \).

3. Results

We consider two parallel wires (see figure 2) with a quadratic cross section of 1.0mm \( \times \) 1.0mm, a length \( l = 20m \), a distance \( d = 3.0mm \) and an electric conductivity of \( \gamma = 5.7 \cdot 10^7 \text{S/m} \) (copper, non-magnetic material). The cross section of the wires is discretised into \( n \times n \) elements, while the the length is discretised into 2 elements. A voltage source with an effective value of \( U^{(0)} = 1.0V \) is applied between the wires. The conductors of the voltage source and the backward shortcut are treated as ideal conductors. The impedance of these two connectors is neglected in our calculations. For proper calculations up to a frequency of \( f = 100 \text{kHz} \) we chose a discretisation of the wire cross section comparable or significantly smaller than the skin depth [3].

In all PGMRES cases we used an early restart of \( m_{max} = 5 \). We resume the results of our investigations:
Table 1: Comparison of computing time at $f = 20$ kHz. Stopping criteria: $5 \cdot 10^4$ (GMRES) and $10^2$ (Gauss-Seidel preconditioner). The last two columns show the maximum error of absolute value and phase of the current density in the intersection of the wires with respect to the Gaussian solver. PGMRES was performed with an early restart of 5.

<table>
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Figure 3: Comparison of convergence history for GMRES and PGMRES (Gauss-Seidel preconditioner, early GMRES restart $m_{\text{max}} = 5$ for a discretisation $n_{\text{disc}} = 14$)

1. In figure 3(a) it is well seen that both unpreconditioned GMRES and PGMRES are convergent. This holds for the whole frequency range (see table 1). For the unpreconditioned GMRES we chose a large restart value of $m_{\text{max}} = 60$ which is only limited by the available memory of our computer.

2. The Gauss-Seidel stopping criteria is choosen similar to the stopping criteria of GMRES (15). A small value $tol$ for the Gauss-Seidel stopping decreases the number of GMRES iterations (see 3(a)) but increases the number of Gauss-Seidel iterations. Overall a further decrease of a small Gauss-Seidel stopping criteria results in an increase of computing time. In our example best results were achieved with a relatively large stopping criteria of $tol = 0.01$.

3. The chosen GMRES preconditioner yields to a convergence acceleration by a factor 8 compared with the unpreconditioned GMRES. Furthermore, from table 1 it can be seen that the GMRES and PGMRES
methods are significantly faster than a direct solver (Gaussian solver).

4. Although at higher frequencies the imaginary part of the impedance matrix will have a large influence on the electric network the good convergence enhancement of the proposed preconditioner is kept in the whole studied frequency range \( f \leq 100 \text{ kHz} \) (see fig.3(b)).

**Conclusion**

The authors simulated inductance problems by the Finite Network Theory (FNM) and presented the applicability of the GMRES and the PGMRES method at a medium frequency range up to 100 kHz.

In the case of a two-dimensional electric network the system matrix of the FNM equation system delivers a Gauss-Seidel preconditioner for GMRES. This preconditioner was derived in the low frequency limit \( f \to 0 \).

Best enhancements of computing time were achieved with an early restart of the preconditioned GMRES method and a relatively large stopping criteria of the Gauss-Seidel solver.

Furthermore, these efforts in computing time were achieved without an acceleration of the matrix-vector products. In our calculation we use the whole dense impedance matrix for matrix-vector products. The implementation of the approximate calculation of these matrix-vector products by hierarchical techniques such as Fast Multipole Method (FMM) [8, 9] should result in an additional acceleration of computing time.

**REFERENCES**