An Eigenvalue Hybrid FEM Formulation for 2D Open Structures Using Mixed type Node/Edge Elements and a Cylindrical Harmonics Expansion

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Abstract

A Finite element formulation for the solution of the two-dimensional eigenvalue problem for open radiating structures is proposed. The semi-infinite solution domain that occurs in such problem is modelled using an expansion in an infinite sum of cylindrical harmonics, while the structure itself is described by the finite element method. The two mathematical models are coupled by exploiting the tangential field continuity condition. In fact for the truncation of the finite element mesh a fictitious cylindrical domain boundary is used which encloses the opening of our structure. On that fictitious boundary we impose the field continuity condition formulating in that way a generalized eigenvalue problem taking in to account Sommerfeld radiation condition. This final eigenvalue problem is solved using the Arnoldi subspace iterative technique, [5].

I. Introduction

The finite element method has proven its robustness in almost all the forms of closed boundary problem. But when it comes to open radiating structures which means that the solution domain extends to infinity, this method cannot be imposed directly. In such cases several extensions of the method have been proposed. The idea behind all these extensions is the implementation of artificial boundaries enclosing the structures, which are transparent to the field solution. This enables the conversion of the unbounded problem to its bounded equivalent. The main representatives of this approach are techniques like the Absorbing Boundary Conditions and the Perfect Matching Layer. Another way to cope with open boundary problems is by considering a field solution satisfying the radiation condition, for the semi-infinite domain beyond the artificial boundary. In turn, this solution is combined with the Finite Element formulation by employing the Field continuity conditions on the artificial boundary. This basic idea had led to a variety of Hybrid methods like the Unimoment [1], and the By-moment method, [2]. In the present effort we proposed a FEM eigenvalue formulation for two-dimensional (2D) open radiating structures similar to the unimoment method. The 2-D solution domain in the generalized case is enclosed by a fictitious circular contour (C). But, for certain shapes of the guiding structures, it is also possible to enclose only the radiating aperture of the structure in an angular sector-C'. For the field solution inside-C the FEM formulation is applied based on a triangular mixed node/edge elements discretization. Thus being able to model arbitrary shaped radiating 2D structures. The field solution in the semi-infinite space outside-C is expressed by an expansion in an infinite series of cylindrical harmonics. The final formulation is obtained by imposing the field continuity conditions on the fictitious surface and exploiting the orthogonality properties of the cylindrical harmonics. The proposed technique preserves the sparsity of the matrix, and the elemental matrices of the final formulation are evaluated analytically. The present work is based on our recent publication,[3], where rectangular mixed node/edge elements were employed. The disadvantage of [3] is the poor discretization of the solution domain, which especially along the circular contour-C causes undesired stair-case effects. Thus the main enhancements of the present approach are the analytical evaluation of the elemental matrices and the use of triangular elements. The use of triangular elements, enables us to discretize accurately the fictitious circular boundary as well as the analytical evaluation of the contour integrals along that.

II. Formulation.

The general topologies of our problem are shown in Fig.1. The two dimensional open boundary structure is enclosed within a circular contour-C with radius $\rho_c$ (Fig 1a). Alternatively the radiating
aperture of the waveguiding structure is enclosed in an angular sector-C’ (Fig 1b). In the second case we consider that the media surrounding the structure below the aperture (region-III) is a perfect electric conductor. So in that way, and for some other reason that will be clarified latter, we can use an “angular sector” domain instead of a complete circular disk. The latter constitutes one of the contributions of the present effort.

Propagation along the longitudinal axis-z as $e^{-jβz}$ is assumed. The structure of Fig.1 is divided into two sub-regions, a bounded region-I inside contour-C (or C’) and the unbounded region-II outside the contour-C (or C’). A vector electric field FEM formulation based on mixed node/edge elements is employed in region-I. For the unbounded semi-infinite domain (region-II) the electric field components can be expressed in terms of $E_z$ and $H_z$ through Maxwell curl equations. These two components ($E_z$ and $H_z$) are in turn expressed as an infinite sum of cylindrical harmonics with unknown weighting factors. The two different description of the field are subsequently joined together by imposing the field continuity conditions on the fictitious boundary C.

The whole procedure followed for the formulation of the generalized eigenvalue problem is explicitly described in our previous work [3]. Namely, starting from the wave equation for the electric field ($\mathbf{E}$), discriminating $\mathbf{E}$ into a transverse and axial component, with respect to the z-axis and applying the Galerkin procedure, a weak FEM formulation is obtained for the region-I. This region is discritized using mixed node/edge triangular elements, instead of the rectangular ones employed in [3]. This enables the accurate discretization of the fictitious contour –C (or C’) as well as the analytical evaluation of the contour integrals along C (or C’). Since, this is a significant contribution of this work it will be further analyzed.

For the unbounded region –II the longitudinal field components for the electric and magnetic fields are assumed to expressed by the expansion bellow [9]:

$$E_z(\rho)\big|_{\rho \geq r_p} = \sum_m \left(A_m^e \cos k\phi + A_m^a \sin k\phi \right) H_m^{(2)}(k,\rho)$$

$$H_z(\rho)\big|_{\rho \geq r_p} = \sum_m \left(B_m^e \cos k\phi + B_m^a \sin k\phi \right) H_m^{(2)}(k,\rho)$$

Where $H_m^{(2)}(k,\rho)$ is the Hankel function of the second kind and order m.

In the above equations the unknown coefficients $A_m^e, A_m^a$ and $B_m^e, B_m^a$ are expressed in terms of the electric field values ($E_z, E^\prime_z$) on the contour-C. This is achieved by enforcing the continuity of the tangential electric field along the C-contour, using the orthogonality properties of the functions $\cos n\phi$ and $\sin n\phi$ and exploiting Maxwell’s curl equations. Due to the space restrictions of the
We will present here the solution, only of some indicating examples of the resulting contour integrals involved in the final elemental matrix for the first order triangular element.

\[
I_1 = \oint_C \frac{\rho}{A^2} \left[ (A_{m/n} + B_{m/n} \rho \sin \phi) (-\sin \phi) + (C_{m/n} - B_{m/n} \rho \cos \phi) \cos \phi \right] d\phi
\]

\[
I_2 = \oint_C \frac{1}{\rho} \cdot \frac{\partial x_i / \partial \phi}{c_i \rho} dl = \oint_C \frac{1}{2A} \cdot \left[ (-b_{i/j} \sin \phi + c_{i/j} \cos \phi) \frac{\partial \phi}{\partial \phi} \right] d\phi
\]

These are now evaluated analytically and their final expressions are obtained as:

\[
I_1 = \frac{\rho L_{mn}}{4A^2} \left[ (-\cos \phi_1 + \cos \phi_2) A_{m/n} + (\phi_1 - \phi_2) \rho B_{m/n} + (-\sin \phi_1 + \sin \phi_2) C_{m/n} \right]
\]

\[
I_2 = \frac{\rho}{2A} \cdot \left[ b_{i/j} (\cos \phi_2 - \cos \phi_1) + c_{i/j} (\sin \phi_2 - \sin \phi_1) \right]
\]

A point of interest concerns the ability of the potential of the cylindrical harmonics expansion to describe the field outside the angular sector contour C’ (Fig.1b). Since the angular sector consisting of perfect electric conductor (PEC) imposes the vanishing of the tangential electric field, then the contour integrals along that is identically zero. Namely, there is no need to impose field continuity on the PEC sector.

Finally the resulting generalized eigenvalue problem is solved using the Arnoldi subspace iterative technique [5], which allows the calculation of a certain eigenmode spectrum.

### III. Numerical Results

The validity of the method was tested against experimental results given by Lambariello et al. [4]. In this reference a leaky waveguide antenna was analyzed and measured. The cross section of the measured antenna, which works in the X-band, is presented in Fig.2a. The mesh used for the eigenvalue calculation is shown in Fig.2b. A complete description of this structure is provided in [4].

![Figure 2](image)

**Figure 2.** The analyzed leaky wave antenna, a) Geometry and b) Triangular mesh discretization (a = b = 11.95mm, a = 23mm, c = 15.65mm and d = 4.55mm, F1 = 21.5mm, F2 = 15mm)
Figure 3. Dispersion curves of the leaky waveguide of Fig. 2, a) Imaginary and b) Real part of the propagation Constant

The dispersion curves obtained from the solution of the generalized eigenvalue problem are shown in Fig.3, for both the imaginary (Fig.3a) and the real (Fig.3b) part. Measurements from [4] are available only for one mode. An excellent agreement is observed between our calculated results (dotted lines) and the experimental results (continuous line), [4]. We also provide another leaky mode captured by the presented technique. More results along with the field distributions will be presented during the conference.

IV. Conclusions

The proposed technique is an elegant way to analyze radiating wave guiding structures. It preserves lot of the FEM advantages as the capability to analyze inhomogeneous and arbitrary shaped structures and preserves the sparsity of the final linear system. Triangular elements are used to avoid “stare case effects” in the discretization of the fictitious contour enclosing the structure. The integrals involved in the definition of the elemental matrices are evaluated analytically. The proposed method provides accurate results for both the real and the imaginary part of the dispersive diagram of any eigenmode, including the complex leaky modes.

A Further improvement of this 2-D method is under consideration where locally curvilinear elements will be employed for the precise modeling of the circular fictitious boundary. Also, the extension to three-dimensional open problems employing a spherical harmonics expansion is considered.

REFERENCES