Constrained Parametric Minimization for the Inversion of Electromagnetic Measurements in the Earth

Aria Abubakar
Schlumberger Doll Research
36 Old Quarry Road
Ridgefield, CT, USA
aabubakar@slb.com

Tarek M Habashy
Schlumberger Doll Research
36 Old Quarry Road
Ridgefield, CT, USA
habashy1@slb.com

Abstract

In this paper we present an algorithm to solve a full nonlinear inverse scattering problem where the parameterization of the model geometry are possible. The algorithm is a constrained Gauss Newton approach in which the regularization parameter is determined automatically in each iterative step. As a numerical example we present the inversion of the electrical conductivity from the low frequency electromagnetic measurements collected in a borehole-logging configuration.

Introduction

We consider a full nonlinear inverse scattering problem where the discrepancy between the measured and predicted data is iteratively minimized in a least-squares sense. With this algorithm we aim on applications where parameterization of the model geometry are possible. By carrying out this parameterization, the number of unknown model parameters that need to be inverted is manageable. Hence, a Newton-based approach can be used advantageously over gradient-based approaches. In order to guarantee an error reduction of the optimization process, the iterative step is adjusted using a line search algorithm. Further the inverted model parameters are constrained using a nonlinear transformation. This constrain forces the reconstruction of the unknown model parameters to lie within their physical bounds. In order to deal with cases where the measurements are redundant or lacking sensitivity to certain model parameters causing non-uniqueness of solution, the cost functional to be minimized is regularized by adding a penalty term. One of the crucial aspects of this approach is how to determine the so-called regularization parameter determining the relative importance of the misfit between the measured and predicted data and the penalty term. Inspired by the work in [1], we show that by choosing the regularization parameter proportional to the normalized misfit between the measured and predicted data we end up with an adaptive procedure for determining the regularization parameter [2]. By combining all the techniques mentioned above we arrive at an effective and robust parametric algorithm.

Inversion Algorithm

We define the vector of residuals \( \bar{e}(\bar{x}) \) as a vector whose \( j \)-th element is the residual error of the \( j \)-th measurement, defined as the difference between the predicted and measured responses: 
\[
e_j(\bar{x}) = S_j(\bar{x}) - m_j, \quad \text{where} \quad j = 1, ..., M.
\]
The symbol \( \bar{x} = [x_i; i = 1, ..., N] \) denoted the unknown model parameter, which is to be obtained by inversion. We pose the inversion as the minimization of the following cost functional of the form:
\[
C(\bar{x}) = \frac{1}{2} \left( \| \bar{W}_d \cdot \bar{e}(\bar{x}) \|^2 - \bar{\chi}^2 + \lambda \| \bar{W}_x \cdot (\bar{x} - \bar{x}_{pr}) \|^2 \right).
\]
The first term of the right hand side of the cost functional drives the inversion towards matching the measured responses (data). The second term of the cost functional is included to regularize the optimization problem. It safeguards against cases when measurements are redundant or lacking sensitivity to certain model parameters causing non-uniqueness of solution. It also suppresses any possible magnification of errors in our parameter estimations due to noise, which is unavoidably present in the measurements. These error magnifications may result in undesirable large variations in the model parameters, which may cause instabilities in the inversion. \( \bar{W}_d' \cdot \bar{W}_d \) is the inverse of the model covariance matrix representing the degree of confidence in the prescribed model parameter.
\( \bar{x}_{\text{pre}} \) and is also provided as \textit{a priori} information. It can also be used to bias certain parts of the model parameter \( \bar{x} \) towards the prescribed model \( \bar{x}_{\text{pre}} \). \( \bar{W}^T_d \cdot \bar{W}_d \) is the inverse of the \textit{data covariance matrix}, which describes the estimated uncertainties due to noise contamination in the available data set. It describes not only the estimated variance for each particular data point, but also the estimated correlation between errors. It therefore provides a point by point weighting of the input data according to a prescribed criterion, hence, can be used to reduce the effect of outliers in the data.

The positive factor \( \lambda \) is the regularization parameter determining the relative importance of the two terms in the cost functional. The determination of \( \lambda \) will produce an estimate of the model parameter \( \bar{x} \) that has a finite minimum weighted norm away from a prescribed model \( \bar{x}_{\text{pre}} \) and which globally misfits the data to within a prescribed value \( \chi \) determined from \textit{a priori estimates} of noise in the data.

Following the analysis in [1], the regularization parameter \( \lambda \) is found as
\[
\lambda = \frac{\| \bar{W}_d \cdot \sigma(\bar{x}_{k-1}) \|^2}{\| \delta \|^2},
\]
where subscript \( k \) denotes the number of iteration and the constant scalar factor \( \delta \) is a parameter related to the normalization of the unknown model parameter.

To impose \textit{a priori} information such as positivity or maximum and minimum bound on the unknown model parameter \( \bar{x} \), we constrained them using the following nonlinear transformation:
\[
x_i = \frac{x_{i_{\text{max}}} + x_{i_{\text{min}}}}{2} + \frac{x_{i_{\text{max}}} - x_{i_{\text{min}}}}{2} \sin(c_i), \quad - \infty < c_i < \infty, \quad i = 1, \ldots, N
\]
where \( x_{i_{\text{min}}} \) and \( x_{i_{\text{max}}} \) are lower and upper bound on the physical model parameter \( x_i \). It is clear that \( x_i \to x_{i_{\text{min}}} \) as \( \sin(c_i) \to -1 \) and \( x_i \to x_{i_{\text{max}}} \) as \( \sin(c_i) \to 1 \).

To obtain a stationary point; \( \bar{x}_k \), where the cost functional attains a minimum, we employ a Gauss-Newton iteration search. The use of this technique is desirable because it partially takes into account the curvature of the cost functional and hence often provides adequate convergence in neighborhood of the minimum. The Gauss-Newton minimization approach is known to have a rate convergence, which is slightly less than quadratic but significantly better than linear. It provides quadratic convergence in the neighborhood of the minimum. Let \( \bar{p}_k \) denotes the Gauss-Newton search direction, and then the unknown model parameter is updates as
\[
\bar{x}_k = \bar{x}_{k-1} + \alpha_k \bar{p}_k,
\]
where
\[
\bar{p}_k = \left[ \mathbf{1}^T \cdot \bar{W}_d \cdot \bar{W}^T_d \cdot \bar{f} + \lambda \bar{W}_x \cdot \bar{W}^T_x \right] \mathbf{1}^T + \lambda \bar{W}_x \cdot \bar{W}^T_x \cdot (\bar{x}_{k-1} - \bar{x}_{\text{pre}}), \quad \bar{f} = \nabla \bar{e}_k
\]
and \( \alpha_k \) is a scalar positive factor which is determined from line minimization along the Gauss-Newton search direction. Note that \( \alpha_k \) is only used in case \( \bar{p}_k \) fails to sufficiently reduce the value of the cost function within two successive iterations.

Finally, we remark that the iteration process will stop if either the relative error \( \sqrt{\| \bar{e}_k \|^2 / M} \) reaches a prescribed value determined from estimates of noise in the data, the differences between two successive iterates of the model parameters are within a prescribed tolerance, or the number of iterations exceeds a prescribed maximum.
Numerical Examples

As an example we consider inversion of non-invaded transverse isotropic (TI) layering formation, which can be parameterized as given in Figure 1.

Figure 1. The layered TI anisotropic formation.

We aim to reconstruct the horizontal formation resistivity \( R_{hi} \), vertical formation resistivity \( R_{vi} \) and the bed boundary \( z_i \) of every layer. Furthermore the dip angle \( \theta \) will also be reconstructed. Hence the number of unknown model parameters is \( N = 3L \) where \( L \) is the number of layers. In order to obtain enough sensitivity to carry out this inversion, the data are assumed to be obtained from triaxial induction measurement, see [2]. This type of measurement incorporates three mutually orthogonal transmitter and receiver coils. Then, for each receiver position we measure nine orthogonal magnetic field components (i.e., an \( x \)-directed transmitter with \( x \), \( y \) and \( z \)-directed receivers, an \( y \)-directed transmitter with \( x \), \( y \) and \( z \)-directed receivers and an \( z \)-directed transmitter with \( x \), \( y \) and \( z \)-directed receivers).

In the inversion we assumed that the azimuth angle is solved first by rotating the data matrix so that the cross coupling \( xy \) (\( x \) transmitter and \( y \) receiver), \( yx \), \( yz \) and \( zy \) are zero. This puts the coordinate \( y \)-axis along the relative strike of the formation. Hence, we have only five components data per transmitter-receiver spacing left to carry out inversion of the remaining parameters. The transmitter and receiver coils are modeled as point’s magnetic dipoles because it has been demonstrated that a point magnetic dipole is accurate at observation distances greater than several coil radii. For each logging point, we recorded the triaxial vector magnetic fields in six different positions with respect to the transmitter. The transmitter and receiver separations are varied from 15 inch up to 72 inch. The frequency operation of the transmitter is 20 kHz.

The configuration we have considered consists of a twenty-eight layer where the well is deviated \( \theta = 30^\circ \). The model has been adapted from the standard Oklahoma formation to replicate TI anisotropy model. The data are collected at 178 logging points, distributed uniformly from \( z = -10 \) up to \( z = 167 \) foot.

The true, initial and inverted models after 15 iterations are given in Figure 2a. In this figure the true model is given by the solid-red-lines while the initial model and the inverted model parameters are given by the dashed-green and dotted-blue lines. We observe that all the unknown model parameters are reconstructed very well. Note that after 15 iterations the square root of the cost function is reduced to 0.0018%.
Further, the inversion results from data corrupted with 3% pseudo random white noise is given in Figure 2b. In Figure 2b we observe only small effect of noise in the reconstructions of the vertical resistivities. Finally one should also note that the value of the cost functional in the inversion of noisy data converge to a certain value which corresponds to the noise level, see Figure 3.

![Figure 2](image.png)

**Figure 2.** Inversion results of the twenty-eight TI anisotropy layers example from data without noise (a) and with 3% random white noise (b). The true model, the initial estimate and the inverted model are given by the solid-red, dashed-green and dotted-blue lines.

![Figure 3](image.png)

**Figure 3.** The square root of the cost functional as function of iteration.

**Conclusion**

In this paper we described an efficient and robust parametric inversion algorithm to retrieve the unknown model parameters from electromagnetic measurement of data. More numerical examples including inversion of full three-dimensional configurations will also be discussed in the presentation.

**REFERENCES**