A Multiplicative Regularized Born Iterative Algorithm

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Abstract
We present a robust iterative method to solve the inverse scattering problem in cases where the Born approximation is valid. We formulate this linearized inverse problem in terms of the unknown material contrast and the unknown contrast sources and we solve the problem by minimizing a cost functional. Further, a multiplicative regularization technique is employed in order to enhance the algorithm. Synthetic numerical examples are presented in order to compare the presented algorithm to the traditional Born algorithm. Results with respect to the inversion of experimental data are presented as well.

Introduction
We consider the microwave tomography problem to determine the location and the electromagnetic composition of a bounded object from measurements of the scattered electromagnetic wavefield, when the object is illuminated by a known wavefield. As starting point to solve this nonlinear and ill-posed problem we take the so-called Multiplicative Regularized Contrast Source Inversion (MR-CSI) method (Van den Berg and Abubakar [1]). When we are dealing with weak scatterers, the Born approximation may be used advantageously. Then, we assume that the total fields inside the scattering objects are approximately equal to the incident fields. The Born approximation reduces the nonlinear problem into a linear problem. In the literature, a large number of linear inversion algorithms are available, but the major problem is that, especially for a limited dataset, even these Born-based linear inversion schemes are not robust. In this paper we present the MR-CSI method when the Born approximation used in it.

Inversion Algorithm
The Multiplicative Regularized Born Inversion (MRBI), see [2], consists of an algorithm to construct sequences \( \{w_j\} \) and \( \{\chi_j\} \) which iteratively reduce the value of the cost functional,

\[
F_n = \left[ F_S + F_{D,n} \right] F_n^R = \left[ \sum_j \left( \frac{\|f_j - G_S w_j\|^2}{\|f_j\|^2} + \frac{\|\chi_j u_j^{\text{inc}} - w_j\|^2}{\|\chi_j u_j^{\text{inc}}\|^2} \right) \right] \frac{1}{V_D} \int_D \left| \nabla \chi_j \right|^2 + \delta_n^2 \, dv
\]

where

\[
\left[ G_S w_j \right](\vec{x}) = \int_D g(\vec{x}, \vec{x}') w_j(\vec{x}') \, dv(\vec{x}')
\]

and the subscripts \( D \) and \( S \) indicate that the observation point \( \vec{x} \) lies either in \( D \), a bounded domain containing the scattering object, or \( S \), a domain disjoint from \( D \) on which the scattered field \( f_j \), \( j = 1, \ldots, J \), is measured for each known incident field \( u_j^{\text{inc}} \). The symbol \( V \) denotes the volume of the domain \( D \). Further, \( \|\cdot\|_S \) and \( \|\cdot\|_D \) denote the norms on \( L_2(S) \) and \( L_2(D) \). Note that the field quantity can be a scalar (for the 2D TM polarization case), a two-components vector (2D TE polarization case), or a three-component vector (3D case). Further, \( g \) denotes the Green function of the background medium, while \( \chi \) denotes the contrast against the background medium. The contrast sources \( w_j \) are related to the incident fields \( u_j^{\text{inc}} \) (Born approximation) by \( w_j = \chi u_j^{\text{inc}} \). By using this approximation, physically it means that we neglect the multiple scattering inside the object \( D \) and
mathematically it means that the Born approximation reduces the nonlinear problem to a linear one. For the steering parameter $\delta_n^2$ we choose progressively decreasing values in such a way that, for given contrast sources, the cost functional $F_n$ as a function of the contrast $\chi$, remains convex during all iterations. We relate this parameter directly to the decreasing object error $F_{D,n-1}$.

The structure of the cost functional is such that it will minimize the regularization factor $F_n^R$ with a large weighting parameter in the beginning of the optimization process, because the value of $F_n + F_{D,n-1}$ is still large, and that it will gradually minimize more and more the error in the data and object equations when the value of $F_n^R$ has reached a nearly constant value equal to one. If noise is present in the data, the data error term $F_S$ will remain at a large value during the optimization and therefore, the weight of the regularization factor will be more significant. Hence, the hindering character of noise will, at all times, be suppressed in the reconstruction process, but at the cost of decreased resolution. This minimization is carried out in two alternate steps. For given contrast, $\chi_{n-1}$, the contrast sources are updated via conjugate gradient directions of the cost functional, while for given contrast sources, $w_{j,n}$, the contrast is updated via a preconditioned conjugate gradient direction of the cost functional.

**Numerical Examples**

First we will compare the method presented in this paper to the standard Born inversion method using a numerical example. Note that all the dimensions are denoted in terms of the wavelength in vacuum. The scattering configuration consists of two distinct squares homogeneous cylinders of diameter $3\lambda/4$ separated by a distance also $3\lambda/4$. The contrast is real and equal to $\chi = 0.5$. This means that the homogeneous cylinders have relative permittivity of $\varepsilon/\varepsilon_0 = 1.5$. These scatterers are located inside a test domain $D$ with sides of length $3\lambda$. In the inversion this test domain $D$ is subdivided into 29 by 29 sub domains. The measurements are carried out on a circle $S$ of radius $3\lambda$, around the center of $D$. The incident fields are chosen to be excited by line sources parallel to the axis of the scatterers. These sources are taken to be equally spaced on the measurement circle. The data are collected on the same circle $S$. We use 29 sources and 29 receivers. In order to compare the method proposed in this paper with traditional Born Inversion algorithm, we define the so-called "error in contrast" at iteration $n$ as

$$\text{ERR}_n = \sqrt{\|\chi_n - \chi_{\text{exact}}\|_D^2/\|\chi_{\text{exact}} + 1\|_D^2}$$

where $\chi_n$ is the complex contrast distribution of the reconstructed profile at iteration $n$ and $\chi_{\text{exact}}$ is the exact one. In Figure 1, we present the error in contrast, $\text{ERR}_n$, as a function of iteration $n$. In Figure 1a we present the error in contrast after the inversion from synthetic data generated, using the Born approximation. The error in contrast, $\text{ERR}_n$, for the standard Born inversion (BI) method is given by the dotted red lines, while the ones for the regularized Born (RBI) method ($F_n^R$ in the cost functional is set to be equal to unity) and for the multiplicative regularized Born inversion (MRBI) method are given by the dashed green and the solid blue lines, respectively. Using the Born data, we commit a kind of "inverse crime" and we observe that all the three inversion methods show almost the same convergence. In Figure 1b we present the error in contrast after the inversion from exact data (data generated by solving the exact forward problem numerically). In this case we observe that the standard Born inversion (BI) method diverges very quickly and hence we conclude that without any appropriate stopping criterion this method is useless. On the other hand the RBI method remains stable and the addition of the multiplicative regularization in the MRBI method improves the results slightly.

Next, in order to experiment with the kind of data we will have in the practical situation, we add also 10% random white noise of the maximum amplitude of all data.
Figure 1. The error in contrast $\text{ERR}_n$ as function of the number of iteration $n$, using the BI method (red dotted lines), the RBI method (green dashed lines) and the MRBI method (solid blue lines), for reconstructions from Born data (a), exact data (b) and exact data with 10% noise (c).

Figure 2. The exact real of the contrast profile (a) and its reconstructed profiles from the exact data with 10% noise, using the BI method (b), the RBI method (c) and the MRBI method (d).

The errors in contrast for all the three inversion methods, using these corrupted data, are presented in Figure 1c. We observe that now the use of the multiplicative regularization factor significantly improves the optimization process. In Figure 2, we present the exact real part of the contrast and its reconstructed profiles from the corrupted exact synthetic data after 1024 iterations, using our three versions of the Born inversion algorithms. From the results in Figure 2, it is obvious that, in order to invert realistic data, we need a reliable scheme such as the RBI method. Furthermore, the inclusion of the multiplicative regularized weighted $L_2(D)$-norm regularizer in the MRBI method enhances the reconstructed profiles significantly.

With regards to the computational time all three methods use almost the same CPU time. The only disadvantage of the RBI and MRBI methods is the extra memory requirement to store the contrast source quantity during the optimization process.

Next we consider the inversion of experimental data conducted using a prototype of the 3D microwave tomographic systems switched to its 2D mode. A detailed description of the measurement setup can be found in [3]. The frequency of operation employed was 900 MHz. The polarization of the electric vector of the emitted electromagnetic radiation was in the vertical direction. Only the vertical component of the electric field is measured. The current prototype allows measurements of attenuation in the tomographic working chamber up to 120 dB with a signal-to-noise ratio of about 30 dB. The phantom object under study was located on a rotating support plate on the axis of the cylindrical tomographic working chamber with a height of 40 cm and a diameter of 60 cm. An automatic motion system and motion-control system place the receiving antennas at various points on the circle of radius 15.5 cm. The phantom was irradiated from 64 directions and the receiving antenna was moved around the phantom on circle with radius of 18.5 cm. It measured the electric field at 38 positions in front of the antenna. Hence in total we have 64 by 38 data points. The details of the phantom under study is given in Figure 3.
Figure 3. Phantom used to generate experimental data. All dimensions are in millimeter (mm).

Figure 4. Reconstructed relative permittivity distribution of the experimental phantom: real part (a) and imaginary part (b).

This phantom has different distances between the details ranging from 8 to 20 mm. The real part of the permittivity is \( \text{Re}(\varepsilon/\varepsilon_0) \approx 70 \pm 5 \), while the imaginary part is \( \text{Im}(\varepsilon/\varepsilon_0) \approx 15 \). The phantom was placed in the center of the tomographic working chamber filled with a salt solution with \( \varepsilon/\varepsilon_0 \approx 77.5 + i15 \). In the inversion we assume that the phantom is entirely located inside the test domain \( D \) of 180 mm by 180 mm. This test domain is divided into 127 by 127 sub domains. The reconstructed profiles after 1024 iterations using the MRBI method are presented in Figure 4. The reconstructed real part is given in Figure 4a and while its imaginary part is given in Figure 4b. The imaginary part is nearly equal to 15, as it should be. We observe that by using only the present linear inversion scheme we are able to retrieve almost all the details of this low contrast phantom. One should note that the present results are similar to the results obtained by a full nonlinear inversion method using a Newton method, which is computationally far more intensive. One iteration of the present method is approximately 3 seconds on Personal Computer with Pentium IV 2.0 Ghz processor. Only the details in the top-left of the phantom are not well retrieved, but the dimensions of these details are less than 20 mm. Since the wavelength in salt solution at 900 MHz is about 33 mm, we surmise that this drawback is caused by the limitation of the resolution of the operating frequency.

REFERENCES