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Abstract
An exact solution is presented for the problem of radiation of an electromagnetic wave from an open end of a circular waveguide cavity formed by a perfectly conducting cylinder and an internal plate termination with non zero impedance. The cavity is excited non symmetrically by the TM dominant mode. This problem has been formulated as a vector diffraction problem for two scalar potentials. The key result of this paper is the correct representation of the unknown scalar potentials in the Fourier transform domain which shows the TM and TE waves interaction at the open end and takes into account the impedance boundary conditions at the termination. These allow to recast the vector diffraction problem as coupled Wiener-Hopf equations with respect to unknown analytical functions in an overlap complex half-planes. In view of the termination a set of infinitely many expansion coefficients is involved. Finely the problem is reduced to infinite system of linear algebraic equations due to the factorisation and decomposition procedure. The method provides a straightforward formulation for the solution and is valid for arbitrary geometrical and frequency parameters.

Introduction
The analysis of electromagnetic scattering by metallic waveguide cavities is an important subject in radar cross section (RCS) reduction and target identification studies. In the previous papers, we have considered several two-dimensional (2-D) cavities formed by a finite parallel-plate waveguide with a planar termination at the open end, and solved the plane wave diffraction rigorously using the Wiener-Hopf technique [1,2]. It has been shown by numerical computation that our results are valid over a broad frequency range and can be used as a reference solution for validating the RCS results obtained by means of more general approximate approaches such as high-frequency and numerical methods. We also developed the Wiener-Hopf technique for rigorous analysis three dimensional (3-D) perfectly conducting circular waveguide cavities [3] as a more realistic model of jet engine intakes of aircrafts and is important in the RCS studies.

Statement of the problem
The mixed boundary value problem for mentioned above wave diffraction by cylindrical waveguide cavity involves the unknown TM and TE scalar potential that satisfy the Helmholtz equation

\[ \frac{\partial^2 u_l}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u_l}{\partial \rho} + \frac{\partial^2 u_l}{\partial z^2} + \left(k^2 - \frac{m^2}{\rho^2}\right) u_l = 0, \quad l = 1, 2, \]  

and boundary conditions

Cylindrical faces
\{ -L < z < L with \( \rho = b - 0 \) and \( \rho = b + 0 \) \}

\[ \frac{\partial^2 u_l}{\partial z^2} + k^2 u_l = 0, \quad -\frac{m}{i \omega \epsilon \rho} \frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial \rho} = 0; \]

Impedance plate termination: \{ 0 < \rho < b with \( z = -L \) \}

\[ \frac{\partial}{\partial \rho} \left( Z_l u_l' - \frac{1}{i \omega \epsilon \rho} \frac{\partial u_l'}{\partial z} \right) + \frac{m}{\rho} \left( u_z'' - \frac{Z_l}{i \omega \epsilon \rho} \frac{\partial u_z'}{\partial z} \right) = 0, \]

\[ m \left( Z_1 u_z' - \frac{1}{i \omega \epsilon \rho} \frac{\partial u_z'}{\partial z} \right) + \frac{\partial}{\partial \rho} \left( u_z'' - \frac{Z_1}{i \omega \epsilon \rho} \frac{\partial u_z'}{\partial z} \right) = 0. \]
Let the total field \( u_1^i(\rho, z) \) be given by

\[
\begin{align*}
\frac{u_1^i(\rho, z)}{u_1(\rho, z)} &= \begin{cases} 
 u_1^i(\rho, z) + u_1(\rho, z), & u_1(\rho, z), \\
 0 < \rho < b & -L \leq z < \infty, \\
 \rho > b & -\infty < z < \infty,
\end{cases}
\end{align*}
\]

where \( u_1^i(\rho, z) \) is the incident TM mode for perfectly conducting infinite cylinder, being defined as:

\[
u_1^i(\rho, z) = e^{imj} J_m(\xi_j \rho / b)e^{-jz'}
\]

with the complex amplitude \( e_{mj} \); \( \xi_j \) for \( j = 1, 2, 3, \ldots \) denote the zeros of Bessel function \( J_m(\cdot) \), \( \gamma_j = (\xi_j / b)^2 - k^2 \). \( \text{Re} \gamma_j > 0 \).

Next we take the Fourier transform of the Helmholtz equation and use the radiation condition.

Applying the method established in our previous papers [1-3], we derive the transformed wave equations as in

\[
\hat{U}_l(\rho, \alpha) = 0, \quad \text{in } \rho > b \text{ for } |\tau| < k_2,
\]

\[
\hat{U}_l(\rho, \alpha) + e^{i\alpha L} \Psi^*_l(\rho, \alpha) = e^{-i\alpha L} \left[ \hat{g}_l(\rho) - i\alpha \hat{f}_l(\rho) \right]
\]

Figure 1. Geometry of the problem.

The terms on the left-hand sides of (5) are the Fourier transforms of the functions appearing in (3), and are defined by

\[
U_1(\rho, \alpha) = \Phi_1(\rho, \alpha) + e^{i\alpha L} \Psi_1(\rho, \alpha) - U_1^i(\rho, \alpha),
\]

\[
U_2(\rho, \alpha) = \Phi_2(\rho, \alpha) + e^{i\alpha L} \Psi_2(\rho, \alpha),
\]

for \( 0 < \rho < b \), where

\[
\Psi_1(\rho, \alpha) = U_1^+(\rho, \alpha) + Q_1(\rho, \alpha), \quad \Psi_2(\rho, \alpha) = U_2^+(\rho, \alpha),
\]

\[
U_1^+(\rho, \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{+L} u_1(\rho, z)e^{i\alpha z} \, dz, \quad \Phi_1(\rho, \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{+L} u_1^i(\rho, z)e^{i\alpha z} \, dz.
\]

Here \( U_1^i(\rho, \alpha) \) and \( Q_1(\rho, \alpha) \) are known functions. In (6)-(8), the subscripts ‘±’ imply that the functions are regular in the half-planes \( \tau > \pm \alpha \), whereas the \( \Phi_1(\rho, \alpha) \) is an entire function.

The unknown inhomogeneous terms are expanded as in

\[
\hat{f}_1(\rho) = f_{10}(\rho / b)^{m_1} + \sum_{n=1}^{\infty} f_{1n} I_m(\xi_n \rho / b), \quad \hat{g}_2(\rho) = g_{20}(\rho / b)^{m_2} + \sum_{n=1}^{\infty} g_{2n} I_m(\eta_n \rho / b),
\]

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Using (11), we find the Fourier transform of the magnetic components
\[ \tilde{f}_2(\rho) = -c_1 \left( \frac{\rho}{b} \right)^m + \frac{Z_{1/0}}{i0} \tilde{g}_2(\rho), \quad \tilde{g}_1(\rho) = -i0 \tilde{c}_1 \left( \frac{\rho}{b} \right)^m + i0 Z_1 \tilde{f}_1(\rho). \] (10)

In (9)-(10), \( f_{in} \) and \( g_{2n} \) for \( n=0,1,2,3,... \) and \( c_1 \) are the unknown coefficients; \( \eta_n \) for \( n=1,2,3,... \) denote the zeros of \( J'_n(\cdot) \), where the prime implies the derivative of the Bessel function with respect to the argument, \( I_m(\cdot) \) is the modified Bessel function of the first kind. Next by following a procedure similar to that developed in [3], we arrive at the solution of (4) and (5) that leads to the scattered field representation in the Fourier transform domain (6) as follows

\[
U_1(\rho, \alpha) = \left\{ \begin{array}{ll}
i0 E_1^+(b, \alpha)e^{i\alpha \tau} \frac{K_m(\gamma \rho)}{\gamma^2} & \text{for } \rho > b, \\
i0 E_1^+(b, \alpha)e^{i\alpha \tau} \frac{I_m(\gamma \rho)}{\gamma^2} & + i0 \frac{Z_1}{i0} \left[ c_1 - (Z_1 - Z \cdot \alpha f_{10}) \tilde{e}_{10} \left( \frac{\rho}{b} \right)^m \right] + (11a) \\
+ ikZ^{-1} \left( Z_k^{-1} \alpha - Z_1 \right) \sum_{n=1}^\infty \frac{f_{in} e^{-i\alpha \tau}}{\alpha^2 + \gamma_n^2} I_m(i \gamma_n \rho / b) - U_1(\rho, \alpha) & \text{for } 0 < \rho < b, \\
\end{array} \right.
\]

\[
U_2(\rho, \alpha) = \left\{ \begin{array}{ll}
\tilde{V}_2^+(\alpha) & \text{for } \rho > b, \\
\tilde{V}_2^+(\alpha) & + \frac{Z_1}{i0} \left( \frac{Z_1}{i0} - 1 \right) \sum_{n=1}^\infty \frac{g_{2n} e^{-i\alpha \tau}}{\alpha^2 + \gamma_n^2} I_m(i \gamma_n \rho / b) & \text{for } 0 < \rho < b, \\
+ \frac{Z_1}{i0} & \sum_{n=1}^\infty \frac{g_{2n} e^{-i\alpha \tau}}{\alpha^2 + \gamma_n^2} I_m(i \gamma_n \rho / b) \\
\end{array} \right. + (11b)
\]

Here \( E_1^+(b, \alpha) \) and \( \tilde{V}_2^+(\alpha) \) are unknown functions regular in the upper half-plane; \( \{ g_{2n} \} \) and \( \{ f_{in} \} \) are two sets of unknown coefficients. Equations (11) hold in the strip \( |\gamma| < k_2 \) and is non-standard because the terms with static multiplier \( (\rho / b)^m \) are involved. In order to ensure the non-dependence of the field components with respect to these static terms it is found, that \( g_{20} = i0 \tilde{c}_1 f_{10} \).

Then the joint unknown coefficients \( c_1, f_{10}, g_{20} \) are also involved in (11).

**Exact solution of the boundary problem.**

Using (11), we find the Fourier transform of the magnetic components \( h_\beta(\rho = b \pm 0, z) \), \( h_\beta(\rho = b \pm 0, z) \) and taking the difference between the resultant equations we derive, that

\[
\begin{align*}
\frac{ik0 E_1^+(b, \alpha)e^{i\alpha \tau}}{\gamma^2 b M_1(\alpha)} & - \frac{m \alpha \tilde{V}_2^+(\alpha)}{(\alpha - k) M_2(\alpha)} - \frac{k}{bZ} (Z_k^{-1} \alpha - Z_1) \sum_{n=1}^\infty \frac{f_{in} e^{-i\alpha \tau}}{\alpha^2 + \gamma_n^2} I_m(i \gamma_n) + \\
+ \frac{m \alpha \tilde{V}_2^+(\alpha)}{i0 \mu b^2} & \sum_{n=1}^\infty \frac{g_{2n} e^{-i\alpha \tau}}{\alpha^2 + \gamma_n^2} I_m(i \gamma_n) + \frac{Z_1}{i0 \mu} \sum_{n=1}^\infty \frac{g_{2n} e^{-i\alpha \tau}}{\alpha^2 + \gamma_n^2} I_m(i \gamma_n) = \frac{\alpha \tilde{z}}{i0 \mu b^2} e^{i\alpha \tau}, \\
\end{align*}
\] (12a)

and

\[
\begin{align*}
\tilde{V}_2^+(\alpha) & + \frac{1}{i0 \mu b^2} \sum_{n=1}^\infty \frac{g_{2n} e^{-i\alpha \tau}}{\alpha^2 + \gamma_n^2} I_m(i \gamma_n) = \tilde{f}_1(\beta, \alpha) e^{i\alpha \tau}, \quad (12b)
\end{align*}
\]

In (12),
\[ M_1(\alpha) = I_m(\gamma b)K_m(\gamma b), \quad M_2(\alpha) = \gamma^2 I'_m(\gamma b)K'_m(\gamma b), \]
\[ a_{mj} = \varepsilon_m^j \xi_j b^{-1} I'_m(\xi_j). \]

Equations (12) are the desired coupled Wiener-Hopf equations hold in the strip \(|\tau|<k_2\).

As has already been shown, \(\Psi_1^+(\rho, \alpha)\) and \(\Psi_2^+(\rho, \alpha)\) are regular in the half-plane \(\tau> -k_2\) while \(\Phi_l(\rho, \alpha)\) \((l=1, 2)\) are entire functions. Therefore it follows that
\[
\lim_{\alpha \rightarrow \eta_n} (\alpha - \gamma_n)[\Phi_1(\rho, \alpha) + \Psi_1^+(\rho, \alpha)e^{\alpha L}] = 0, \quad (15a)
\]
\[
\lim_{\alpha \rightarrow \eta_n} (\alpha - \gamma_n)[\Phi_2(\rho, \alpha) + \Psi_2^+(\rho, \alpha)e^{\alpha L}] = 0, \quad (15b)
\]

for \(n=0,1,2,3, \ldots; \gamma_0 = \tilde{\gamma}_0 = -ik\). From these arrive to the relations
\[
\tilde{V}_1^+(k) = -\frac{1}{2iZ} [c_1 - (Z_1 - Z)f_{10}]e^{-ikl}, \quad \tilde{V}_2^+(k) = -\frac{m}{2kb} [(ZZ_1 - 1)g_{20} - ikc_1] e^{-ikl}, \quad (16a)
\]
\[
\tilde{V}_1^+(i\eta_n) = \frac{kb^2}{2Z\eta_n} [i\eta_n - k] I'_m(i\eta_n)e^{\eta_n L} f_{1n}, \quad (16b)
\]
\[
\tilde{V}_2^+(i\eta_n) = \frac{b}{2} \left( i\eta_n \frac{Z_1}{\eta_n} - 1 \right) I'_m(i\eta_n)e^{\eta_n L} g_{2n}, \quad (16c)
\]

where \(\tilde{V}_1^+(\alpha) = i\omega(\alpha + k)^{-1} E_1^+(b, \alpha)e^{\alpha L}\).

Next we apply the factorization and decomposition procedure for solution of the Wiener-Hopf equations (12). This leads to the infinite linear algebraic system, that can be solve for arbitrary geometrical parameters and frequency with pre-specify accuracy. We also derive the field representation and analyse the particular cases numerically.

Conclusions

An exact solution of new problem for electromagnetic wave radiation from an open end of a circular waveguide cavity are presented here. The cavity is formed by a perfectly conducting cylinder and an internal plate termination with non zero impedance and exciting non symmetrically by TM dominant mode. This solution has wide range of application in particular can be used for design a new approaches as well as a benchmark for the comparison of approximate technique applied to more practical problems.

The key result of this paper is the correct analytical representation of the unknown scalar potentials in the Fourier transform domain (11) which shows the TM and TE waves interaction at the open end and takes into account the impedance boundary conditions at the termination. This leads to the coupled Wiener-Hopf equations. Finally, the problem is reduced to infinite system of linear algebraic equations due to the factorisation and decomposition procedure. The method provides a straightforward formulation for the solution and is valid for arbitrary geometrical and frequency parameters. The mode field representation for the cavity region and far field pattern using the saddle point technique for integration are also received. The closed form solution are received for the case of non symmetrically excitation by the TM dominant mode of the semi-infinite perfectly conducting cylinder without termination.

REFERENCES

