Diffraction by a Set of Three Parallel Impedance Half-Planes with the One Amidst Located in the Opposite Direction

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Abstract

The diffraction problem of plane waves by a set of three parallel half-planes with different surface impedances on upper and lower faces where the one amidst is placed in the opposite direction, is solved by Mode-Matching method where available and Fourier transform technique elsewhere. The solution includes two independent Wiener-Hopf equations each involving infinite number of expansion coefficients which satisfy an infinite system of linear algebraic equations.

Introduction

Diffraction of acoustic and electromagnetic waves by a set of parallel half-planes is an important problem in scattering theory and has been subjected to numerous past investigations. The case of two half-planes lying parallel to each other with edges in a plane normal to the half-planes with either Dirichlet or Neumann boundary conditions on all faces of the half-planes has been treated extensively in the literature [1], [2] and [3]. Due to the symmetry of the configuration with respect to the plane amidst, the two-dimensional Wiener-Hopf formulation allows an elementary decoupling and the reduction to two scalar Wiener-Hopf equations. The case with Dirichlet conditions on one half-plane (both faces) and Neumann conditions on the other requires the factorization of a 2×2 matrix of the Daniele-Khrapkov form. The solution was given by Hurd and Lüneburg in 1981 [4].

Three parallel equally spaced non-staggered rigid half-planes have been first investigated by Jones [5], who formulated the problem as a three-dimensional matrix Wiener-Hopf equation. This equation is first reduced to a system of 2×2 Wiener-Hopf equations and a single scalar one. Recently Buyukaksoy and Polat [6] and Alkumru[7] have treated the diffraction of plane waves by two and three parallel impedance half-planes, respectively in the case where the half-planes have certain thicknesses. The analysis in all the above mentioned cases requires the solution of an infinite system of algebraic equations.

Meister, Rottbrand and Speck [8] considered the diffraction by n-parallel half planes characterized by first, second and third-kind boundary conditions and discussed the solvability of the resulting 2N×2N matrix Wiener-Hopf equations.

An important class of diffraction problems consists of the scattering of plane waves by three parallel impedance half-planes is the situation where one of which is placed in opposite direction with respect to the others. Recently Buyukaksoy and Cinar [9] has considered the case where the half-plane which is located in the opposite direction is the bottom one. The aim of this work is to treat the case where the half-plane amidst is placed in the opposite direction. For the sake of generality we will assume that the upper and lower faces of the half-planes are characterized by constant but different surface impedances. Instead of a classical formulation which leads to a matrix Wiener-Hopf equation, the total field in the waveguide region is expressed in terms of the normal modes while the Fourier transform technique is used elsewhere. Hence, the related boundary value problem is formulated as two independent Wiener-Hopf equations whose solutions require determination of infinitely many expansion coefficients satisfying an infinite system of linear algebraic equations. The effect of the surface impedances and the spacings between the half-planes on the diffraction phenomenon is presented.

A time factor $e^{\text{i}\omega t}$ with $\omega$ being the angular frequency is assumed and suppressed throughout the paper.

The Solution of the Problem

Let an $E_z$-polarized plane wave
where \( k = 2\pi/\lambda \) (\( \lambda \) being the wave-length) is the free space propagation constant and \( \varphi_o \) is the angle of incidence, illuminate the structure formed by three parallel impedance half-planes with normalized surface impedances \( \eta_j^+ \) (\( j = 1, 2, 3 \) where \( \eta_j = Z_j/Z_o \) with \( Z_o \) being the free-space characteristic impedance). Here, the superscript + and – indicate upper and lower surfaces, respectively.

Figure 1. Geometry of the diffraction problem.

In order to avoid the presence of a matrix Wiener-Hopf equation during the formulation, the field in the waveguide region can be expressed in terms of normal modes. Hence for analysis purposes, it is convenient to express the total field, say \( u_t(x,y) \), as follows:

\[
u_t(x,y) = \begin{cases} 
  u_t^i(x,y) + u_t^i(x,y) + u_1(x,y) & , \ y > b \\
  u_2^{(1)}(x,y)H(-x) + u_2^{(2)}(x,y)H(x), & 0 < y < b \\
  u_2^{(3)}(x,y)H(-x) + u_2^{(4)}(x,y)H(x), & -b < y < 0 \\
  u_3(x,y) & , \ y < -b 
\end{cases}
\]

(2)

with \( u_t(x,y) \) being the incident field given in (1). Here, \( H(x) \) stands for the unit step function while \( u_r(x,y) \) is the field reflected from the plane with normalized surface impedance \( \eta_1^+ \), namely:

\[
u_r(x,y) = \frac{\eta_1^+ \sin \varphi_o - 1}{\eta_1^+ \sin \varphi_o + 1} \exp \left\{ -ik \left[ x \cos \varphi_o - (y - 2b) \sin \varphi_o \right] \right\}
\]

(3)

The boundary conditions and the continuity relations, together with the edge and radiation conditions help the solution of the Helmholtz equations for the field expressions. Applying the Fourier transform technique

\[
F_\pm(\alpha, y) = \mp \int_0^{\pm\infty} u_1(x,y) e^{i\alpha x} dx \quad H_\pm(\alpha, y) = \pm \int_0^{\infty} u_3(x,y) e^{i\alpha x} dx
\]

(4)

and considering

\[
P_\pm(\alpha) = F_\pm(\alpha, b) + \frac{\eta_1^+}{ik} F'_\pm(\alpha, b) \quad Q_\pm(\alpha) = H_\pm(\alpha, -b) - \frac{\eta_2^+}{ik} H'_\pm(\alpha, -b)
\]

(5)

the mixed boundary-value problem is reduced to two decoupled modified Wiener-Hopf equations which are

\[
K(\alpha) \frac{\eta_1^+}{\chi \eta_3^+ \alpha} P_\pm(\alpha, b) + F'_\pm(\alpha, b) = \frac{2k \sin \varphi_o}{\eta_1^+ \sin \varphi_o + 1} \left( \alpha - k \cos \varphi_o \right) \\
\ \\
\sum_{m=1}^{\infty} \left( f_m - i\alpha g_m \right) K_m = \left( \cos K_m b + \frac{\eta_2^+}{ik} K_m \sin K_m b \right)
\]

(6)

and
\[ K(\alpha) \frac{\chi(\eta_1, \alpha)}{\chi(\eta_1, \alpha)} Q_+ (\alpha) + H_+ (\alpha, -b) = \sum_{j=1}^{\infty} \left( \frac{p_j - i\alpha q_j}{\alpha^2 - \gamma_j^2} \right) \gamma_j \left( \cos \Gamma_j b + \frac{\eta_1^-}{ik} \Gamma_j \sin \Gamma_j b \right) \]  

(7)

Here, the terms \( \chi(\eta, \alpha) \), \( N^{(1)}(\alpha) \) and \( N^{(2)}(\alpha) \) stand for

\[ \chi(\eta, \alpha) = \frac{K(\alpha)}{\eta K(\alpha) + k} \quad N^{(1,2)}(\alpha) = e^{ikb} M^{(1,2)}(\alpha) \]  

(8.a,b)

with

\[ M^{(1)}(\alpha) = \frac{\left( \eta_1^+ - \eta_1^- \right)}{ik} K \cos Kb + \left( 1 - \frac{\eta_1^-}{k^2} K^2 \right) \sin Kb \]  

(9)

\[ M^{(2)}(\alpha) = \frac{\left( \eta_1^- - \eta_2^- \right)}{ik} K \cos Kb - \left( 1 - \frac{\eta_2^-}{k^2} K^2 \right) \sin Kb \]  

(10)

In equations (6) and (7), the coefficients \( f_m \), \( g_m \), \( p_j \) and \( q_j \) are solved through

\[ P_+ (\alpha_m) = -\left( f_m - i\alpha_m g_m \right) \frac{d}{d\alpha} \left[ M^{(1)}(\alpha) \right]_{\alpha = \alpha_m} \]  

(11)

\[ Q_+ (\gamma_j) = \frac{p_j - q_j}{\Gamma_j} \left( \cos \Gamma_j b - \frac{\eta_2^-}{ik} \Gamma_j \sin \Gamma_j b \right) \]  

(12)

with

\[ \gamma_n = \frac{\Gamma_j}{2\gamma_j} \left( \cos \Gamma_j b + \frac{\eta_2^-}{ik} \Gamma_j \sin \Gamma_j b \right) \frac{d}{d\alpha} \left[ M^{(2)}(\alpha) \right]_{\alpha = \gamma_j} . \]  

(13)

In the equations (6), (7), (11), (12), and (13) \( \alpha_m \)'s and \( \gamma_j \)'s stand for the infinite number of symmetrical zeros of functions \( M^{(1)}(\alpha) \) and \( M^{(2)}(\alpha) \), respectively, satisfying

\[ M^{(1)}(\pm \alpha_m) = 0 \quad M^{(2)}(\pm \gamma_m) = 0 \quad \Im(\alpha_m, \gamma_m) > \Im(k) \quad m = 1, 2, \ldots \]  

(14)

The classical Wiener-Hopf procedures for each decoupled equation yield

\[ \sqrt{k + \alpha} \frac{\chi(\eta_1^+, \alpha)}{\chi(\eta_1^+, \alpha)} P_+ (\alpha) = \frac{2k \sin \varphi_o}{\chi(\eta_1^+, \alpha)} \left[ \frac{\chi(\eta_1^+, k \cos \varphi_o)}{\chi(\eta_1^+, \sin \varphi_o + 1)} \frac{N^{(1)}(k \cos \varphi_o)}{N^{(1)}(k \cos \varphi_o)} \right] \]  

(15)

\[ -\sum_{m=1}^{\infty} \left( f_m + i\alpha_m g_m \right) K_m \left( \cos K_m b + \frac{\eta_1^+}{ik} K_m \sin K_m b \right) \frac{\chi(\eta_1^+, \alpha_m)}{\chi(\eta_1^+, \alpha_m)} N^{(1)}(\alpha_m) \]  

and

\[ Q_+ (\gamma_j) = \frac{\chi(\eta_1^-, \alpha)}{\chi(\eta_1^-, \alpha)} N^{(2)}(\alpha) \sum_{j=1}^{\infty} \left( \frac{p_j + i\gamma_j q_j}{\gamma_j} \right) \gamma_j \left( \cos \Gamma_j b + \frac{\eta_1^-}{ik} \Gamma_j \sin \Gamma_j b \right) \frac{\chi(\eta_1^-, \gamma_j)}{\chi(\eta_1^-, \gamma_j)} N^{(2)}(\gamma_j) \]  

(16)

The diffracted field is determined through

\[ u_1(x, y) = \frac{1}{2\pi} \int_k \frac{\chi(\eta_1, \alpha)}{K(\alpha)} P_+ (\alpha) e^{ik(x \cos \theta - y \sin \theta)} d\alpha . \]  

(17)

In the above equation, the change of variables \( x = \rho \cos \varphi \), \( y = \rho \sin \varphi \) and \( \alpha = -k \cos \varphi \) and application of the saddle-point technique yield
A computational analysis is also made to observe the effects of the surface impedances and the distance between the half-planes on the diffracted and transmitted fields. Two of the graphical results are shown below.

Figure 2. The effect of the surface impedance $Z_1^+$ on the diffracted field.

Figure 3. The effect of $b$ on the diffracted field.

It is also shown that in the case when $b$ tends to zero, the result yields nothing but the well-known solution related to the diffraction by two-part impedance plane given by [10].

References


