Weierstrass’ Solutions to Certain Nonlinear Wave and Evolution Equations

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Abstract

A method is presented for finding a subset of the exact (traveling-wave) solutions of various nonlinear wave and evolution equations (NLWEE). By using an appropriate ansatz $\psi \rightarrow f$ the NLWEE is transformed into an ordinary differential equation $(f')^2 = P$, where $P$ is (in general) a fourth degree polynomial in $f$. The solutions of this differential equation are expressed compactly in terms of Weierstrass’s elliptic function $\wp$ and include (real, bounded) periodic and solitary-wave-like solutions.

The approach applied can be outlined as follows. The starting point is a NLWEE which describes a certain dynamical system by means of a (wave) function $\psi (x,t)$. An appropriate transformation $\psi \rightarrow \bar{f}$, where $\bar{f}$ is supposed to obey the nonlinear differential equation

$$\left( \frac{df(x)}{dx} \right)^2 = \alpha f^4 + 4\beta f^3 + 6\gamma f^2 + 4\delta f + \varepsilon = R(f) \quad (1a)$$

(with real $x$, $f(x)$, $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$) transforms the NLWEE into an equation $P(\bar{f})=0$, where $P$ denotes a polynomial in $f$. Vanishing coefficients in the polynomial equation $P(\bar{f})=0$ lead to equations which partly determine the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ in Eq. (1a). In general, the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ depend on the structure and parameters of the NLWEE and, finally, on the parameters of the transformation of $\psi$.

Thus, the problem of finding a solution of the NLWEE is reduced to a solution of Eq. (1a), which, in this sense, is the basis of the following analysis (for reference purposes we shall call it the "basic equation" of the associated NLWEE). The higher derivatives $f_{xx}$, $f_{xxx}$, ..., appearing in the analysis of the various NLWEEs, can easily be obtained from the basic equation,

$$f_{xx} = 2\alpha f^3 + 6\beta f^2 + 6\gamma f + 2\delta, \quad (1b)$$

$$f_{xxx} = (6\alpha f^2 + 12\beta f + 6\gamma)f_x, \quad (1c)$$

The solution of the basic equation (1a) has been well known since 1865. It reads
\[ f(x) = f_0 + \sqrt{R(f_0)} \frac{d\varphi(x; g_2, g_3)}{dx} + \frac{1}{2} R'(f_0) \left[ \varphi(x; g_2, g_3) - \frac{1}{24} R''(f_0) \right] + \frac{1}{24} R(f_0) \frac{1}{24} R'''(f_0) \]

\[ 2\left[ \varphi(x; g_2, g_3) - \frac{1}{24} R''(f_0) \right]^2 - \frac{1}{48} R(f_0) \frac{1}{24} R'''(f_0) \]

where the primes denote differentiation with respect to \( f \) and \( f_0 \) is any constant, not necessarily a real root of \( R(f) \). The invariants \( g_2, g_3 \) of Weierstrass’s elliptic function \( \varphi(x; g_2, g_3) \) are related to the coefficients of \( R(f) \) by

\[ g_2 = \alpha \varepsilon - 4\beta \delta + 3\gamma^2, \]

\[ g_3 = \alpha \gamma \varepsilon + 2\beta \gamma \delta - \alpha \delta^2 - \gamma^3 - \varepsilon \beta^2. \]

It should be noted that \( \varphi \) is real if \( x \) is real.

If there exists a simple root \( f_0 \) of \( R(f) \), Eq.(2) can be simplified to

\[ f(x) = f_0 + \frac{R'(f_0)}{4\left[ \varphi(x; g_2, g_3) - \frac{1}{24} R''(f_0) \right]}. \]

In most cases considered in the following, Eq.(5) is sufficient to find analytical solutions of the NLWEEs. Nevertheless, Eq.(2) is useful if there is no simple root of \( R(f) \) (e.g., for kink solitary waves), or if it is difficult to identify the simple root appropriate for real bounded solutions \( f(x) \). The behavior of \( f(x) \) can be classified by the quantity

\[ \Delta = g_3^3 - 27g_2^2 \]

which is the discriminant of the Weierstrass’s function \( \varphi \) as well as of \( R(f) \). This fact opens the possibility to combine algebraic and geometric arguments to study the properties of the basic equation \( f_x^2 = R(f) \) and thus the properties of \( f(x) \). According to the (double) periodicity of \( \varphi \) (in Eqs. (2) and (5)) the real period of \( f(x) \) can be finite or infinite, corresponding to periodic or solitary wave like solutions, respectively. In particular, the latter are associated with

\[ \Delta = 0, \quad g_2 \geq 0, \quad g_3 \leq 0. \]

Physical solutions \( f(x) \) must be real and bounded. Thus, \( f_x^2 \) must be nonnegative and bounded. The graph of \( R(f) \) shows that \( f(x) \) will vary monotonically until \( f_x \) is zero. Hence it follows that the real roots of \( R(f) \) are important for studying the properties of \( f(x) \). Real bounded solutions are related to the existence of a finite interval \([f_1, f_2]\) where \( R(f) \) is nonnegative and bounded in \([f_1, f_2]\) with real roots \( f_1, f_2 \). We designate this condition “phase diagram condition” of \( f \) (PDC).

If \( \Delta \neq 0 \), the real period of \( \varphi \) is finite. In particular, if \( \Delta > 0 \), all roots of \( R \) are real and simple or two pairs of complex conjugate roots exist. If \( \Delta < 0 \), a pair of real roots of \( R \) and a pair of complex conjugate roots exist. Thus, \( f(x) \) is periodic in these cases. If \( \Delta = 0 \), \( g_3 > 0 \), \( f(x) \) is also periodic (cf. Eq. (8b)). Subject to these conditions and the PDC, Eq.(5) represents the periodic (Weierstrass) solutions of the NLWEE in question.

In order to find explicit analytical expressions for solutions of the NLWEE, Eq.(2) or Eq.(5)
must be evaluated. If an appropriate (with respect to PDC) \( f_0 \) in Eq.(2) or an appropriate simple root \( f_0 \) in Eq.(5) has been identified, the evaluation is straightforward. In particular, if \( \Delta = 0 \), \( \varphi (x; g_2, g_3) \) is given by

\[
\varphi (x;12e_1^2,-8e_1^3) = e_1 + \frac{3e_1}{\sinh^2(\sqrt{3}e_1x)}, \quad g_2 > 0, \quad g_3 < 0, \quad e_1 = \frac{1}{2}\sqrt{g_3}, \quad (8a)
\]

\[
\varphi (x;12e_1^2,8e_1^3) = -e_1 + \frac{3e_1}{\sin^2(\sqrt{3}e_1x)}, \quad g_2 > 0, \quad g_3 > 0, \quad e_1 = \frac{1}{2}\sqrt{g_3}, \quad (8b)
\]

\[
\varphi (x;0,0) = \frac{1}{x^2}, \quad \Delta = g_2 = g_3 = 0. \quad (8c)
\]

Assuming \( f_0 \) is a simple root of \( R(f) \) if \( g_3 \neq 0 \), consistent with PDC, Eq.(5) yields

\[
f(x) = f_0 + \frac{R'(f_0)}{4[e_1 - \frac{R''(f_0)}{24} + 3e_1 \cosh^2(\sqrt{3}e_1x)]}, \quad g_3 < 0, \quad (9a)
\]

\[
f(x) = f_0 + \frac{R'(f_0)}{4[-e_1 - \frac{R''(f_0)}{24} + 3e_1 \cos^2(\sqrt{3}e_1x)]}, \quad g_3 > 0, \quad (9b)
\]

\[
f(x) = f_0 + \frac{6R'(f_0)x^2}{24 - R''(f_0)x^2}, \quad g_2 = g_3 = 0, \quad R(f_0) > 0. \quad (9c)
\]

According to (7), Eqs.(9a) and (9c) represent solitary waves. Eq.(9b) represents a periodic solution of Eq.(1a). If there is no simple root of \( R(f) \), Eqs.(8a) and (8b) must be inserted into Eq.(2) to yield the appropriate analytical solutions. In this case, \( f_0 \) must be chosen according to the PDC such that \( R(f_0) > 0 \).

The approach can be summarized as follows: if the NLWEE can be reduced by a transformation \( \psi \rightarrow f \) to the basic equation (1a) with \( \{\alpha, \beta, \gamma, \delta, \varepsilon\} \neq \{0,0,0,0,0\} \), the solutions of Eq.(1a), and thus of the NLWEE, can be represented in terms of the Weierstrass’s elliptic function \( \varphi (x; g_2, g_3) \), where the invariants \( g_2, g_3 \) can be expressed by the coefficients of \( R \). The phase diagram condition (PDC) determines physical (real and bounded) solutions \( \psi \) (periodic or solitary wave like). Conditions for periodic solutions are given by

\[
\Delta \neq 0 \quad \text{and} \quad \Delta = 0, \quad g_2 > 0, \quad g_3 > 0. \quad (10)
\]

The conditions

\[
\Delta = 0, \quad g_2 \geq 0, \quad g_3 \leq 0. \quad (11)
\]

determine solitary wave like solutions, represented by hyperbolic functions as limiting cases of the Weierstrass’s elliptic function.
It is remarkable, that for many nonlinear wave and evolution equations (e.g. nonlinear Schrödinger, Korteweg-de Vries, Kadomtzev-Petviashvili, Burgers, Korteweg-de Vries-Burgers, Boussinesq, Ginzburg-Landau, sine-Gordon, Double sine-Gordon, higher order nonlinear Schrödinger, generalized compound Korteweg-de Vries, Joseph-Egri, Korteweg-de Vries-Zakharov-Kusnetzov, Kodoma-Hasegava) Weierstrass’ solutions can be obtained by the approach outlined above.